

QUASI-NEWTONIAN DUST COSMOLOGIES

HENK VAN ELST* & GEORGE F R ELLIS†

*Department of Mathematics and Applied Mathematics, University of Cape Town, Rondebosch 7701
Cape Town, South Africa*

May 6, 1998

Abstract

Exact dynamical equations for a generic dust matter source field in a cosmological context are formulated with respect to a non-comoving Newtonian-like timelike reference congruence and investigated for internal consistency. On the basis of a lapse function N (the relativistic acceleration scalar potential) which evolves along the reference congruence according to $\dot{N} = \alpha \Theta N$ ($\alpha = \text{const}$), we find that consistency of the quasi-Newtonian dynamical equations is not attained at the first derivative level. We then proceed to show that a self-consistent set can be obtained by linearising the dynamical equations about a (non-comoving) FLRW background. In this case, on properly accounting for the first-order momentum density relating to the non-relativistic peculiar motion of the matter, additional source terms arise in the evolution and constraint equations describing small-amplitude energy density fluctuations that do not appear in similar gravitational instability scenarios in the standard literature.

PACS number(s): 04.20.-q, 98.80.Hw, 95.30.Sf, 98.65.Dx

*e-mail: henk@gmunu.mth.uct.ac.za

†e-mail: ellis@maths.uct.ac.za

1 Introduction

The aim of this paper is to investigate the Newtonian limit of the Theory of General Relativity in a cosmological context, when conditions relating to gravitational physics are significantly different from the more traditional quasi-stationary and asymptotically flat situations where the Newtonian and post-Newtonian limits can usually be derived and are subject to precise experimental tests. The importance of this is that many astrophysical calculations on the formation of large-scale structure in the Universe rely on such a limit, because most of these calculations are done in a Newtonian way. A number of derivations of such a limit are given in the literature, see, e.g., Refs. [36], [4] and [27]; indeed, there is a viewpoint that cosmology is essentially a Newtonian affair, with the relativistic theory only needed for examination of some observational relations. However:

(A) The true classical theory of gravitation is General Relativity; Newtonian theory is only a good theory of gravitation when it is a good approximation to the results obtained from General Relativity;

(B) We have to extend standard Newtonian theory (which strictly can only deal with quasi-stationary isolated systems in an asymptotically flat spacetime) in some way or another to deal with a non-stationary cosmological context. This extension needs to be clearly spelt out.

Significant questions remain in regard to both points. These include:

(1) Full General Relativity theory involves *ten gravitational potentials* (combined in a tensorial variable), subject to the ten Einstein field equations (‘EFE’), but Newtonian theory involves only one (a scalar variable), subject to the one Poisson field equation (‘PFE’); how does it arise that the other nine potentials and equations can be ignored in the Newtonian limit? Given that nine equations of the full theory are not satisfied even in some limiting sense, how do we know when Newtonian cosmological solutions correspond to consistent relativistic solutions of the full set of equations? Part of the answer is that the coordinate freedom of General Relativity accounts for four of these potentials; however, this still leaves another five to account for, and we have examples of Newtonian solutions with *no* General Relativity analogues [15, 20]. Consequently we need to be concerned as to how well standard Newtonian theory represents the results of General Relativity in the cosmological context we have in mind, when we take a ‘Newtonian limit’.

(2) The issue of *boundary conditions* for Newtonian theory in the cosmological context is problematic even in the context of exactly spatially homogeneous (and spatially isotropic) cosmological models [26]; no fully adequate theory exists in the more realistic almost-homogeneous case. Numerical simulations, for example, usually rely on periodic boundary conditions, which correspond to the real Universe only if we live in a ‘small universe’ [16] in which there is a long-wavelength cut-off in the spectrum of inhomogeneities of its large-scale structure. Analytic solutions usually rely on asymptotically flat conditions which are manifestly not true in a realistic, almost-Friedmann–Lemaître–Robertson–Walker (‘FLRW’) situation (they implicitly or explicitly assume that inhomogeneities die off sufficiently far away from the region of interest).¹

(3) How do we attain a *unique propagation equation* for the gravitational scalar potential in a Newtonian cosmology, when Newtonian theory proper has no such equation? In standard Newtonian theory, which aims to describe only quasi-stationary settings, this results from the assumed asymptotically flat condition — which does not hold in the cosmological context.

(4) How do we satisfactorily handle the *gauge-dependence* that underlies most derivations of a Newtonian limit? Equivalently, most derivations of the Newtonian limit are highly coordinate dependent, despite the strongly proclaimed doctrine of covariance of General Relativity; do we need to revise our view on this?²

One approach to these issues is the ‘frame theory’ suggested by Ehlers [8, 9], which has the advantage of giving precise theorems on some Newtonian limits, but has the disadvantage of being more complex than General Relativity. Here we seek a more direct approach to the Newtonian limit in a cosmological situation, that is more readily related to the approaches taken in the astrophysical context. We do so by introducing a class of exact relativistic cosmologies that we can justifiably call *quasi-Newtonian* because of the properties we discuss below, and which include as special cases linearised models used in many astrophysical studies, for example in gravitational lensing theory.³ In studying these spacetimes, we will

¹Ideally one would impose conditions at a finite distance rather than infinity [13]; however, this ‘finite infinity’ approach has not yet been developed fully.

²See, e.g., the discussion by Ellis and Matravers [19].

³They are related to the spacetimes given this name by Bardeen [3] when we linearise our models about a non-comoving

come across each of the above issues, in a context where we can pose — but not often solve — precise questions concerning exact quasi-Newtonian solutions of the EFE; consequently we will present some unanswered ‘*Problems*’ in what follows — well-posed questions that should be answerable, but are of such complexity that we have been able to make only little progress in their solution. We then examine the effect of linearising about a (*non*-comoving) FLRW model, when these questions become tractable and interesting issues occur in relation to the approaches usually taken.

In order to limit the complexity of the problem, we will only consider cosmological models in which the source of spacetime curvature is given by a pressure-free matter field (‘dust’); that is, we will examine here only the pure gravitational problem. A further feature of our discussion will be the absence of gravitational radiation; on the one hand, this is a property of gravitational fields unknown to the Newtonian theory, and on the other, gravitational radiation is commonly thought of as not being of immediate dynamical relevance in structure formation processes. We proceed as follows: in section 2 we develop our concept of a quasi-Newtonian dust cosmology in $1 + 3$ covariant terms and touch upon issues of the uniqueness of the non-comoving time slicing chosen. In section 3 we present the relevant $1 + 3$ covariant dynamical equations. These are derived from the Ricci and Bianchi identities on using the EFE to algebraically relate the Ricci curvature to the dust matter source field. We then fix the time-reparameterisation freedom of the EFE by proposing to evolve the lapse function along the timelike reference congruence in a way which includes various choices made in the literature. In section 4, with the given choice of lapse function evolution, we pursue a consistency analysis of the dynamical equations to the first derivative level; no consistency is attained at this stage. A number of dynamically specialised subcases are then highlighted in section 5. In section 6 we investigate the effect of consistently linearising our dynamical equations about a non-comoving FLRW background. Compared to results in the standard literature, our approach, which fully accounts for the first-order momentum density of non-relativistically moving matter, leads to modifications in the set of evolution and constraint equations. Our conclusions are contained in section 7. Useful computational details such as commutation relations, the set of $1 + 3$ covariant dynamical equations cast into an associated $1 + 3$ orthonormal frame form, and the linearised EFE for a scalar-perturbed, spatially flat FLRW model have been gathered in an appendix. Throughout our work we will employ geometrised units characterised by $c = 1 = 8\pi G/c^2$. Some of our notation is inspired by Ref. [30].

Our discussion concentrates on two main issues: (i) the internal consistency of the set of *exact* equations defining relativistic quasi-Newtonian dust cosmologies; in looking at this, we need to ask whether or not the time slicing condition we impose leads to a *uniquely* defined foliation \mathcal{T} : $\{t = \text{const}\}$, and have to face the problem of time-reparameterisation freedom of the dynamical equations of General Relativity (the EFE); and (ii) how, if at all, do the *exact* equations reduce under FLRW-linearisation to the Newtonian form encountered in many works on gravitational instability of matter in non-relativistic motion, i.e., what happens to the constraints in this limit, and when is the set of equations so obtained both consistent and linearisation-stable? Of most direct relevance to an astrophysically oriented reader is the discussion given in section 6, where, in relating our approach to some of those given in the astrophysical literature, we show that extra terms not always included seem to be needed if the Newtonian-like equations used are to be consistently derived from General Relativity. The reader may directly address this material after becoming acquainted with the definitions and relations provided in sections 2 and 3.

2 Quasi-Newtonian cosmologies

2.1 Definition

We define relativistic *quasi-Newtonian* cosmologies to be spacetime manifolds $(\mathcal{M}, \mathbf{g})$, subject to the EFE, which contain a congruence of worldlines that is *irrotational* and *shearfree*. We call such a congruence a *Newtonian-like timelike congruence*, and denote the associated future-directed, normalised, tangent vector field by \mathbf{n} ($n_a n^a = -1$), the cosmologies by $(\mathcal{M}, \mathbf{g}, \mathbf{n})$.

Most cosmologies will *not* admit such a congruence of Newtonian-like worldlines, and, hence, are not quasi-Newtonian; examples that *are*, are the FLRW cosmologies.⁴ If a cosmology *does* admit such a congruence, it will almost always be *unique* because it has a special relation to the Weyl curvature of $(\mathcal{M}, \mathbf{g})$; we explore this further below. Because it is irrotational, this congruence is normal to a preferred

FLRW geometry.

⁴And, in the non-cosmological context, the interior and exterior Schwarzschild spacetimes.

family of spacelike 3-surfaces of constant time, $\mathcal{T}: \{t = \text{const}\}$, corresponding to the spacelike 3-surfaces of constant absolute time in a Newtonian spacetime. Because of the shearfree condition, these 3-surfaces are mapped conformally onto each other⁵ by the preferred timelike curves \mathbf{n} . The matter in such a setting will almost always *not* be comoving with these curves (or, equivalently, will not be at rest in the preferred spacelike 3-surfaces). The relativistic acceleration of these curves can be derived from a scalar potential that corresponds to the Newtonian gravitational scalar potential.

In more detail: we aim to describe the physics of a pressure-free matter field from the perspective of a Newtonian-like class of Eulerian observers comoving with \mathbf{n} . To this end, we apply a 1 + 3 covariant treatment adapted to \mathbf{n} (cf. Refs. [7, 12, 21, 30]) rather than the essentially equivalent ADM 3 + 1 formulation of the dynamics of relativistic spacetime geometries [2, 45]. The family of spacelike 3-surfaces $\mathcal{T}: \{t = \text{const}\}$ orthogonal to the timelike reference congruence constitutes the rest 3-spaces of the Eulerian observers, with (time-dependent) 3-metric $h_{ab} := g_{ab} + n_a n_b$ satisfying $\dot{h}_{\langle ab \rangle} = 0 = D_a h_{bc}$. As to notation, for any geometrically defined variable \mathbf{T} , we write $\dot{T}^{a\dots b\dots} := n^c \nabla_c T^{a\dots b\dots}$ and $D_a T^{b\dots c\dots} := h^d_a h^b_e \dots h^f_c \dots \nabla_d T^{e\dots f\dots}$, while angle brackets stand for a fully symmetric tracefree \mathbf{n} -orthogonal projection as in, e.g., $\dot{h}_{\langle ab \rangle} = [h^c_{(a} h^d_{b)} - \frac{1}{3} h_{ab} h^{cd}] \dot{h}_{cd}$. The timelike normals \mathbf{n} to the $\mathcal{T}: \{t = \text{const}\}$ are irrotational:

$$\omega^a(\mathbf{n}) = 0, \quad (1)$$

and shearfree:

$$\sigma_{ab}(\mathbf{n}) = 0. \quad (2)$$

The assumption (2) corresponds to the vanishing of the anisotropic part of the extrinsic curvature of the orthogonal spacelike 3-surfaces $\mathcal{T}: \{t = \text{const}\}$ (these 3-surfaces exist because of property (1)). The setting so outlined shares the features of the ‘zero-shear hypersurfaces’ gauge conditions introduced by Bardeen in his well-known work on linearised gauge-invariant cosmological perturbations [3].⁶

2.1.1 Zero magnetic Weyl curvature

Through the 1 + 3 covariant Ricci identities for \mathbf{n} [12, 21], the kinematical restrictions given by Eqs. (1) and (2) immediately lead to zero *magnetic* Weyl curvature of $(\mathcal{M}, \mathbf{g}, \mathbf{n})$ as seen by the Eulerian observers comoving with \mathbf{n} :

$$H_{ab}(\mathbf{n}) = 0. \quad (3)$$

This has the important implication: there can exist *no* gravitational radiation in a quasi-Newtonian cosmology [24]. Being an invariant physical feature, it must also hold with respect to all other observers *different* from the Eulerian ones comoving with \mathbf{n} . This shows that, considered as General Relativity spacetimes, such cosmological models are very special, for in fact generic motion of matter *will* generate gravitational radiation. However, its absence, of course, corresponds to the situation encountered in the Newtonian theory of gravitation.

2.1.2 Acceleration scalar potential

Preservation of the irrotationality condition (1) requires from the vorticity evolution equation (cf. Refs. [12] and [21]) that $0 = \epsilon^{abc} D_b \dot{n}_c$;⁷ the spatial rotation of the relativistic acceleration $\dot{n}^a(\mathbf{n}) := n^b \nabla_b n^a$ of the timelike reference congruence \mathbf{n} must vanish. Consequently, the acceleration is proportional to the spatial gradient of a 1 + 3 covariantly defined scalar field N [45]:

$$\dot{n}_a(\mathbf{n}) := N^{-1} D_a N. \quad (4)$$

As already mentioned, this relativistic acceleration scalar potential corresponds to the Newtonian gravitational scalar potential. For convenience one may alternatively introduce a dimensionless scalar potential $\Phi := \ln(N/N_0)$, where N_0 is a constant.⁸

⁵They are not mapped *isometrically* onto each other, primarily because we allow for an expansion of their normals. One could make a case that *Strictly Newtonian-Like* solutions should demand this stronger condition; however, that would exclude most of the expanding cosmological models that are a prime concern of this paper (but also are not strictly Newtonian). Accordingly, we stick to the weaker definition given above.

⁶The normals to the $\mathcal{T}: \{t = \text{const}\}$ are also non-shearing for Bardeen’s ‘longitudinal’ gauge conditions [3].

⁷In the *comoving* irrotational case of a barotropic perfect fluid, where the fluid acceleration is proportional to the spatial gradient of the total energy density, this is an identity. Here, however, it provides a *genuine* constraint on \dot{n}^a .

⁸The physical dimension of the lapse function N is [length].

Taken together, the physical properties expressed by Eqs. (1) - (4) clearly justify the terminology ‘quasi-Newtonian’ for the cosmological models at hand.⁹

2.2 Local coordinates

The relativistic acceleration scalar potential N is nothing but the prominent lapse function of the ADM 3 + 1 formalism, which embodies (part of) one’s personal choice of how to ‘move forward in time’ [33] within a dynamical spacetime setting. In terms of a set of (dimensionless) local coordinates $\{x^i\}$, the infinitesimal line element of a quasi-Newtonian cosmology can be written in the form (see, e.g., Ref. [40])¹⁰

$$ds^2 = -N^2(x^i) dt^2 + S^2(x^i) d\sigma^2, \quad d\sigma^2 := f_{\alpha\beta}(x^\gamma) dx^\alpha dx^\beta, \quad n^a = N^{-1} \delta^a_0. \quad (5)$$

(the time coordinate is $t = x^0$; spacetime coordinate indices i, j, k, \dots run through 0 – 3, and spatial coordinate indices $\alpha, \beta, \gamma, \dots$ run through 1 – 3), where the rate of expansion scalar Θ of the normals \mathbf{n} is given by

$$\Theta(x^i) = 3 \frac{\dot{S}}{S} = 3 (SN)^{-1} \frac{\partial S}{\partial t}, \quad (6)$$

and S , as usual, denotes a representative expansion length scale. Because this form of the line element implies that \mathbf{n} is irrotational and shearfree, *existence of local coordinates such that the line element has the form (5) is a necessary and sufficient condition that a cosmology is quasi-Newtonian in our sense* (note that this form includes both the FLRW and Schwarzschild spacetimes).

2.3 The matter flow

The dust matter flow of our cosmological model¹¹ is moving with average 4-velocity $\tilde{\mathbf{u}}$ ($\tilde{u}_a \tilde{u}^a = -1$). Thus, relative to the Lagrangean reference frame comoving with $\tilde{\mathbf{u}}$ (where $\tilde{u}^a(\tilde{\mathbf{u}}) := \tilde{u}^b \nabla_b \tilde{u}^a = 0$), the energy-momentum-stress tensor takes the familiar form

$$T_{ab}(\tilde{\mathbf{u}}) = \tilde{\mu} \tilde{u}_a \tilde{u}_b. \quad (7)$$

1 + 3-decomposing $\tilde{\mathbf{u}}$ with respect to the Eulerian \mathbf{n} -frame we obtain [23, 6]

$$\tilde{u}^a := \gamma (n^a + v^a), \quad \gamma := (1 - v^2)^{-1/2}, \quad v^2 := v_a v^a \geq 0, \quad v_a n^a = 0. \quad (8)$$

The variable v^a is the *peculiar velocity* of the matter flow with respect to the \mathcal{T} : $\{t = \text{const}\}$. Hence, the 1 + 3 covariant decomposition relative to the Eulerian \mathbf{n} -frame (where $U^a_b := -n^a n_b$, $h^a_b = \delta^a_b - U^a_b$) leads to

$$T_{ab}(\mathbf{n}) = \mu n_a n_b + 2 q_{(a} n_{b)} + p h_{ab} + \pi_{ab}; \quad (9)$$

the matter variables are given by [23]¹²

$$\mu = \gamma^2 \tilde{\mu}, \quad p = \frac{1}{3} \mu v^2, \quad q^a = \mu v^a, \quad \pi_{ab} = \mu v_{(a} v_{b)}. \quad (10)$$

Note that p and π_{ab} , which are typically non-Newtonian sources of spacetime curvature, are *quadratic* in v^a , so they will be negligible when we linearise our models about a non-comoving FLRW background, assuming that v^a itself is a deviation variable of first-order smallness. However, q^a is *linear* in v^a ; this will be crucial later on. The dependence of the fluid kinematical variables $\tilde{\sigma}_{ab}$, $\tilde{\Theta}$ and $\tilde{\omega}_{ab}$ on the set $\{n^a, \dot{n}^a, \Theta, v^a, D_a v_b\}$ is given explicitly in subsection A.3 of the appendix.¹³ In the limit $v^a \rightarrow 0$ the situation at hand reduces to the standard comoving description of FLRW cosmological models with dust matter source field (in this limit, the acceleration of the normals has to vanish, because the dust flow lines are geodesic).

⁹For further discussion see Ref. [3], and the discussion of ‘Newtonian-like’ spacetimes in Ref. [32].

¹⁰In the case when $S = S(t)$, we can alternatively use the conformal time coordinate $\eta(t) = \int_{t_0}^t dy/S(y)$ of physical dimension $[\text{length}]^{-1}$, integrated along the integral curves of the timelike reference congruence \mathbf{n} , and the line element takes the form $ds^2 = S^2(\eta)(-N^2(x^i) d\eta^2 + d\sigma^2)$. Note, however, this does not work in general when $S = S(t, x^\alpha)$; for then $\eta = \eta(t, x^\alpha)$ and there will be time-space cross terms in the resultant form of the line element.

¹¹In our discussion, for simplicity, we set the cosmological term to zero, $\Lambda = 0$.

¹²We could consider several matter components: each would obey similar equations, with the total matter variables simply being the sum of those for each component. Note the direct relation to the approach taken in formulating thermodynamical properties of matter within kinetic theory.

¹³For the covariant derivatives of n^a and v^a one obtains, respectively, $\nabla_a n_b = -n_a \dot{n}_b + \frac{1}{3} \Theta h_{ab}$ and $\nabla_a v_b = -(v_c \dot{n}^c) n_a n_b - n_a \dot{v}_{(b)} + \frac{1}{3} \Theta v_a n_b + D_a v_b$.

2.4 Uniqueness of N and \mathbf{n}

As is well-known, the ADM 3 + 1 formalism does *not* provide one with a natural evolution equation for the lapse function N (nor the shift vector N^a), and, therefore, by Eq. (4), also *not* for the acceleration \dot{n}^a of the normal congruence \mathbf{n} [2, 45]. Hence, it is a major dynamical issue as to how to determine, for a given physical problem, the best prescription for moving forward in coordinate time t from one spacelike 3-surface within the foliation to the next [45]. In general, a ‘time slicing condition’ leading to a typically differential constraint relation has to be introduced by hand.¹⁴ However, a fundamental point has already been raised above: despite this lack of uniqueness of N from a dynamical viewpoint, we expect the normal congruence \mathbf{n} in our case to be highly restricted *geometrically*, indeed to usually be uniquely determined, because of the conditions (a) that Eulerian observers comoving with \mathbf{n} will see zero magnetic Weyl curvature, Eq. (3), and (b) have a highly restricted electric Weyl curvature, or, equivalently, a very restricted intrinsic curvature allowed for the spacelike 3-surfaces in the family \mathcal{T} : $\{t = \text{const}\}$ (this will be explored below). The hope then is that choice of this congruence essentially determines the time evolution of N , because it determines the acceleration \dot{n}^a (which occurs in Eq. (4)); so uniqueness of \mathbf{n} will imply a unique choice of the lapse function.

Consequently, it is important to determine when the Newtonian-like timelike reference congruence \mathbf{n} really *is* unique. As regards point (a) above, if we express the Weyl curvature relative to \mathbf{n} in terms of its electric part, we obtain

$$C^{ab}{}_{cd}(\mathbf{n}) = 4n^{[a}n_{[c}E^{b]}{}_{d]} + 4h^{[a}{}_{[c}E^{b]}{}_{d]} . \quad (11)$$

Hence, the magnetic Weyl curvature relative to *another* timelike congruence, $\tilde{\mathbf{n}}$, where $\tilde{n}^a = \gamma(n^a + v^a)$ (cf. Eq. (8)), is

$$\tilde{H}_{ab}(\tilde{\mathbf{n}}) = 2\gamma^2 \epsilon_{cd\langle a} [n_{b\rangle} (E^c{}_e v^e) + E^c{}_{b\rangle}] v^d . \quad (12)$$

So, generically, for $E_{ab} \neq 0$ and $v^a \neq 0$, we have $\tilde{H}_{ab} \neq 0$. On the other hand, both $H_{ab}(\mathbf{n}) = 0$ and $\tilde{H}_{ab}(\tilde{\mathbf{n}}) = 0$, iff

$$\epsilon_{cd\langle a} [n_{b\rangle} (E^c{}_e v^e) + E^c{}_{b\rangle}] v^d = 0 . \quad (13)$$

This will be satisfied in the FLRW case, where $0 = E_{ab}(\mathbf{n}) = H_{ab}(\mathbf{n})$ (implying $0 = \tilde{E}_{ab}(\tilde{\mathbf{n}}) = \tilde{H}_{ab}(\tilde{\mathbf{n}})$; cf. subsection A.3 of the appendix), and also for cosmologies with even shearing timelike reference congruences that possess LRS symmetry, if v^a characterises a Lorentz-boost along the 1 + 3 covariantly defined preferred spacelike direction \mathbf{e} [11, 22]. Equation (13) is a necessary (but not sufficient) condition that the same spacetime geometry can appear Newtonian-like relative to *two* (or more) timelike congruences. When this is satisfied, one then also needs to find if (b) is satisfied: does the electric Weyl curvature have the highly restricted form that allows a zero initial rate of shear to remain zero, given the matter source fields present? These conditions are rather complex, nevertheless, we would like to get a complete solution:

Problem 1: Determine all quasi-Newtonian cosmologies that allow more than one Newtonian-like timelike congruence.

This family includes constant curvature spacetimes and the FLRW spacetimes; are there any others?

3 1 + 3 covariant dynamical equations

Insisting on the timelike reference congruence \mathbf{n} to have zero rate of shear, $\sigma_{ab} = 0$, converts the original $\dot{\sigma}^{(ab)}$ -equation (a Ricci identity determining the evolution of σ_{ab} along \mathbf{n}) into a constraint equation which we can regard as *either* restricting the allowed time slicing, given the Weyl curvature of $(\mathcal{M}, \mathbf{g}, \mathbf{n})$ (i.e., as a differential time slicing condition for the lapse function N), *or* as restricting the allowed electric Weyl curvature, $E_{ab}(\mathbf{n})$, given a choice of the time slicing of $(\mathcal{M}, \mathbf{g}, \mathbf{n})$. In either case, the constraint so generated assumes the explicit form¹⁵

$$0 = E^{ab} - D^{(a}\dot{n}^{b)} - \dot{n}^{(a}\dot{n}^{b)} - \frac{1}{2}\mu v^{(a}v^{b)} . \quad (14)$$

¹⁴This indeterminacy does not arise when the congruence of preferred worldlines is comoving with well-specified matter (e.g., a perfect fluid with specified equation of state). However, that is not the case here.

¹⁵We draw the attention of the reader to the fact that the differential time slicing condition constituted by Eq. (14) is conceptually analogous to obtaining a relativistic generalisation of the PFE when imposing ‘maximal’ time slicing characterised by $\Theta = 0 \Rightarrow \dot{\Theta} = 0$ (but $\sigma_{ab} \neq 0$) in the case of asymptotically flat spacetimes [45].

This is a highly restrictive assumption; we wish to explore its implications.

Using the variables specified in Eq. (10), under the stated assumptions (1) - (4) characterising quasi-Newtonian cosmologies with dust matter source field one derives from the general 1 + 3 covariant dynamical equations (cf. Refs. [12] and [21]) the set:

Evolution equations:

$$\dot{\Theta} = -\frac{1}{3}\Theta^2 + D_a \dot{n}^a + (\dot{n}_a \dot{n}^a) - \frac{1}{2}(1 + v^2)\mu \quad (15)$$

$$\dot{\mu} = -(1 + \frac{1}{3}v^2)\Theta\mu - v^a D_a \mu - \mu D_a v^a - 2\mu(v_a \dot{n}^a) \quad (16)$$

$$\dot{v}^{(a)} = -\frac{1}{3}(1 - v^2)\Theta v^a - v^b D_b v^a + (v_b \dot{n}^b) v^a - \dot{n}^a \quad (17)$$

$$\begin{aligned} \dot{E}^{(ab)} = & -\Theta E^{ab} + \frac{2}{3}(1 - \frac{1}{4}v^2)\Theta\mu v^{(a} v^{b)} + \frac{1}{2}(v^c D_c \mu) v^{(a} v^{b)} - \frac{1}{2}v^{(a} D^{b)}\mu \\ & + \frac{5}{6}(D_c v^c)\mu v^{(a} v^{b)} - \frac{1}{2}\mu D^{(a} v^{b)} + \mu v_c v^{(a} D^{(b)} v^{c)} + \mu v_c v^{(a} D^{[b} v^{c]} \end{aligned} \quad (18)$$

The evolution equation for v^a is a consequence of the original $\dot{q}^{(a)}$ -equation (Bianchi identity); it establishes conservation of momentum (density). As such it is the relativistic analogue of the Euler equation in the Newtonian theory.

Primary constraint equations:

$$0 = (C_1)^a := D^a \Theta - \frac{3}{2}\mu v^a \quad (19)$$

$$0 = (C_2)^a := \epsilon^{abc} D_b \dot{n}_c \quad (20)$$

$$0 = (C_3)^{ab} := E^{ab} - D^{(a} \dot{n}^{b)} - \dot{n}^{(a} \dot{n}^{b)} - \frac{1}{2}\mu v^{(a} v^{b)} \quad (21)$$

$$\begin{aligned} 0 = (C_4)^a := & D_b E^{ab} - \frac{1}{3}(1 + \frac{1}{2}v^2)D^a \mu + \frac{1}{3}[\Theta + \frac{3}{2}\mu^{-1}(v^b D_b \mu) + \frac{5}{3}(D_b v^b)]\mu v^a \\ & + \frac{1}{6}\mu v_b [D^{(a} v^{b)} - 5D^{[a} v^{b]}] \end{aligned} \quad (22)$$

The constraint $(C_1)^a$, being the reduced form of the original $D_b \sigma^{ab}$ -constraint (Ricci identity), states that the peculiar motion of the matter as quantified by v^a induces a non-zero spatial gradient in the expansion rate Θ of the normals \mathbf{n} . Constraints $(C_2)^a$ and $(C_3)^{ab}$ are the key new constraints resulting from the quasi-Newtonian assumptions, $(C_2)^a$ leading to the existence of the relativistic acceleration scalar potential N (see Eq. (4)). $(C_4)^a$ is the $D_b E^{ab}$ -constraint (Bianchi identity). Additionally, there are constraints related to H_{ab} :

$$0 = (C_H)^{ab} := H^{ab} \quad (23)$$

$$0 = (C_{D \cdot H})^a := D_b (C_H)^{ab} + \frac{1}{2}\epsilon^{abc} D_b (\mu v_c) \quad (24)$$

$(C_H)^{ab}$ holds as long as Eqs. (1) and (2) apply. Using Eq. (96) of subsection A.1 in the appendix, the $(C_{D \cdot H})^a$ -constraint (Bianchi identity) can be reduced to $0 = D_b (C_H)^{ab} - \frac{1}{3}\epsilon^{abc} D_b (C_1)_c$ and thus is identically satisfied if each of $(C_1)^a$ and $(C_H)^{ab}$ is valid. It then follows that

$$\epsilon^{abc} D_b v_c = \epsilon^{abc} v_b \frac{D_c \mu}{\mu} \Leftrightarrow D_{[a} v_{b]} = \mu^{-1} v_{[a} D_{b]} \mu \quad (25)$$

the spatial rotation of v^a is equal to the algebraic cross product between v^a itself and the fractional total energy density gradient $(D_a \mu)/\mu$. As such, the vorticity of v^a will be of *second*-order smallness when we only consider first-order deviations about a FLRW background setting associated with the \mathcal{T} : $\{t = \text{const}\}$. Altogether, the fully orthogonally projected covariant derivative of v^a can now be decomposed as

$$D_a v_b = D_{\langle a} v_{b\rangle} + \frac{1}{3}(D_c v^c) h_{ab} + \mu^{-1} v_{[a} D_{b]} \mu \quad (26)$$

$D_{\langle a} v_{b\rangle}$ and $D_a v^a$ representing the distortion (or shear) and spatial divergence (or expansion) of the peculiar velocity field, respectively. With condition (23) prescribed, the status of the $\dot{H}^{(ab)}$ -equation (Bianchi identity) can take either the rank of yet another constraint relation (cf. Ref. [24]), or it naturally yields the propagation of $(C_H)^{ab}$ along \mathbf{n} . Here we find that

$$(C_H)^{ab} = (\dot{C}_H)^{(ab)} = -\Theta(C_H)^{ab} - [D^{(a} + \dot{n}^{(a}](C_2)^{b)} - \epsilon^{cd(a}[D_c + \dot{n}_c](C_3)^{b)}_d \quad (27)$$

if $(C_2)^a$ and $(C_3)^{ab}$ are satisfied on $\mathcal{T}_0: \{t_0 = \text{const}\}$, then a consistent set of initial data for quasi-Newtonian cosmologies with dust matter source field characterised by Eqs. (1) - (4) generates *no* magnetic Weyl curvature along \mathbf{n} .

A particularly important restriction on $(\mathcal{M}, \mathbf{g}, \mathbf{n})$ results as a compatibility requirement between $(C_3)^{ab}$ and $(C_4)^a$. It follows that

$$0 = D_b(C_3)^{ab} = (C_4)^a - \dot{n}_b(C_3)^{ab} - \frac{2}{3}(C_5)^a, \quad (28)$$

where $(C_5)^a$ is given by

$$\begin{aligned} (C_5)^a := & D^a [D_b \dot{n}^b + (\dot{n}_b \dot{n}^b)] + \dot{n}^a [D_b \dot{n}^b + (\dot{n}_b \dot{n}^b) - \frac{1}{3}\Theta^2 + (1 - \frac{1}{2}v^2)\mu] \\ & - \frac{1}{2}(1 + v^2)D^a \mu + \frac{1}{2}[\Theta + 3\mu^{-1}(v^b D_b \mu) + \frac{10}{3}(D_b v^b) + 3(v_b \dot{n}^b)]\mu v^a \\ & + \frac{1}{2}\mu v_b [D^{(a} v^{b)} - 5D^{[a} v^{b]}], \end{aligned} \quad (29)$$

and $0 = (C_5)^a$ is demanded. Using Eq. (4), this provides a third-order partial differential equation for N .

In summary, the particular non-trivial constraints characterising the quasi-Newtonian case at hand are $(C_3)^{ab}$ and its immediate consequence $(C_5)^a$, for these embody the consequences of the zero-shear requirement (2).

3.1 The geometry of the orthogonal spacelike 3-surfaces

Next, we give the geometrical variables characterising the intrinsic curvature properties of the orthogonal spacelike 3-surfaces $\mathcal{T}: \{t = \text{const}\}$, which exist because of the requirement (1).

Trace and tracefree parts of Gauß embedding equation:

$$0 = (C_G) := {}^3R + \frac{2}{3}\Theta^2 - 2\mu \quad (30)$$

$$0 = (C_G)^{ab} := {}^3S^{ab} - E^{ab} - \frac{1}{2}\mu v^{(a} v^{b)}. \quad (31)$$

3R and ${}^3S_{ab}$ denote the respective irreducible parts of the 3-Ricci curvature. As demonstrated for the example of a Lagrangean description of an irrotational dust matter source field in Ref. [41], the Gauß embedding equation has to be included in order to obtain a set of (1 + 3 covariant) dynamical equations which is equivalent to the EFE. Note that the geometrical structure of Eq. (30) is identical to the Friedmann equation in FLRW cosmology.

3-Cotton-York tensor [23]:

$$\begin{aligned} {}^3C^{ab} &:= h^{1/3} \epsilon^{cd(a} D_c {}^3S^{b)}_d \\ &= -2h^{1/3} \epsilon^{cd(a} [\dot{n}_c D^{(b)} \dot{n}_{d)} + \frac{1}{2}\mu v_c (D^{(b)} v_{d)} - D^{[b} v_{d]})] - h^{1/3} \mu v^{(a} \epsilon^{b)cd} \dot{n}_c v_d \\ &\quad + h^{1/3} \epsilon^{cd(a} [D_c (C_G)^{b)}_d + D_c (C_3)^{b)}_d - \dot{n}_c (C_3)^{b)}_d] + h^{1/3} \dot{n}^{(a} (C_2)^{b)}. \end{aligned} \quad (32)$$

The 3-Cotton-York tensor, although *not* having the dimension of a curvature variable, encodes the conformal curvature properties of the $\mathcal{T}: \{t = \text{const}\}$. They are conformally flat if ${}^3C_{ab}$ is zero. It has been suggested that a non-zero 3-Cotton-York tensor could be intimately related to the presence of gravitational radiation in a spacetime geometry (see, e.g., Ref. [42]). Consequently, one would expect it to vanish in the quasi-Newtonian case.

3.2 Propagation of N and \dot{n}^a

Having now attained the full set of 1 + 3 covariant dynamical equations for quasi-Newtonian cosmologies with dust matter source field, it is clear (in accordance with the ADM 3 + 1 results) that there is no explicit equation determining the evolution of N or \dot{n}^a . Two differential expressions containing the latter particularly spring to the eye. In terms of N the \dot{n}^a -part of Eq. (14) is given by

$$D_{\langle a} \dot{n}_{b \rangle} + \dot{n}_{\langle a} \dot{n}_{b \rangle} = N^{-1} D_{\langle a} D_{b \rangle} N. \quad (33)$$

The symmetric tracefree derivative $D_{\langle a} \dot{n}_{b \rangle}$ is the distortion of \dot{n}^a . On the other hand, the spatial divergence $D_a \dot{n}^a$ can be interpreted as its expansion. Again, in terms of N we have

$$D_a \dot{n}^a + (\dot{n}_a \dot{n}^a) = N^{-1} D_a D^a N ; \quad (34)$$

this is a source term in Eq. (15) and corresponds to the Laplacian of the gravitational scalar potential in the Newtonian theory. We also have no evolution equations for neither of these two differential quantities. This remains true if one writes out the full set of equations in an associated $1 + 3$ orthonormal frame formalism, which we give in subsection A.4 of the appendix. Hence, we face:

Problem 2: In those cases where the Newtonian-like timelike congruence is unique, how is the corresponding evolution of the lapse function N along these worldlines determined?

Lacking explicit evolution equations for any of the relativistic acceleration,¹⁶ its spatial divergence,¹⁷ or other spatial derivatives, we presume the former must be implicit in the consistency conditions that arise when we impose the quasi-Newtonian restrictions (1) and (2) on our timelike reference congruence. However, we have been unable to obtain a general prescription as to how this happens.

3.2.1 An Ansatz

In a formal sense, an evolution equation for \dot{n}^a in the quasi-Newtonian dust case can be derived by combining Eq. (4) with the commutator relation (92) in subsection A.1 of the appendix, thus yielding

$$(\dot{n})^{\cdot \langle a \rangle} = N^{-1} D^a (\dot{N}) - \frac{1}{3} \Theta \dot{n}^a . \quad (35)$$

The latter makes explicit the necessity for an evolution equation for N , required to close the set of dynamical equations. Failing any other way of determining this, we propose to evolve N along \mathbf{n} according to the *Ansatz*

$$\dot{N} = \alpha \Theta N \quad \text{where} \quad \alpha = \text{const} , \quad (36)$$

which is a common choice when modelling dynamical spacetime geometries by numerical methods [5]; for $\alpha = 1$, e.g., this Ansatz specialises to the so-called ‘harmonic gauge’ where N propagates along \mathbf{n} via a wave equation at the speed of light (cf. Refs. [1] and [5]). Other particular values for the parameter α will be mentioned in section 6 below. With this assumption, it follows that

$$(\dot{n})^{\cdot \langle a \rangle} = (\alpha - \frac{1}{3}) \Theta \dot{n}^a + \alpha \frac{3}{2} \mu v^a + \alpha (C_1)^a . \quad (37)$$

Thus, the evolution of the dynamical quasi-Newtonian cosmologies is now deterministic.

4 Constraint analysis

To check whether quasi-Newtonian cosmologies with dust matter source field can provide consistent solutions of the EFE requires a detailed analysis of the preservation properties of the constraint relations as a model evolves. That all constraints involved be preserved along the integral curves of \mathbf{n} is a *necessary* condition for consistency of the $1 + 3$ covariant dynamical equations; a *full* consistency check involves additionally showing, e.g., that all Jacobi and Ricci equations of an associated $1 + 3$ orthonormal frame formalism are satisfied (see, e.g., Refs. [11] and [23]), or casting the evolution equations into a first-order symmetric hyperbolic form (see, e.g., Refs. [1] and [25] for the cases of a vacuum and a perfect fluid spacetime geometry, respectively).

¹⁶This contrasts the case of a barotropic perfect fluid represented in a *comoving* frame, where (i) the relativistic acceleration is determined from the spatial gradient of the pressure by the momentum conservation equation, (ii) the pressure is determined from the total energy density via the equation of state, and (iii) the energy density time evolution is determined by the energy conservation equation. In the present situation, because the reference frame is *not* comoving, step (i) fails; given any specified acceleration distribution, the momentum conservation equation determines the peculiar velocity of the matter fluid relative to the Eulerian reference frame.

¹⁷In the *strictly* Newtonian case we would additionally have $\Theta = 0$, and then the Raychaudhuri equation (15) becomes the relativistic generalisation of the PFE determining this spatial divergence. In the case of an expanding cosmological model, however, this equation rather determines the time rate of change of the expansion that is compatible with whatever value holds for this spatial divergence.

As mentioned, ideally we would like to derive the form of \dot{N} in the generic case from this consistency analysis, but have been unable to do so; consequently, a prescription for \dot{N} is needed, followed by a subsequent consistency check. We assume the evolutionary behaviour of the lapse function N along the integral curves of \mathbf{n} to be given by the Ansatz (36). Using relations for the propagation of spatial derivative terms given in Ref. [21]¹⁸ and also Eq. (96) in subsection A.1 of the appendix, we then find that the primary constraints (19) - (22) and (29) propagate according to

$$(\dot{C}_1)^{\langle a \rangle} = -\Theta (C_1)^a + (C_5)^a \quad (38)$$

$$(\dot{C}_2)^{\langle a \rangle} = (\alpha - \frac{2}{3}) \Theta (C_2)^a + \frac{1}{3} \epsilon^{abc} \dot{n}_b (C_1)_c \quad (39)$$

$$\begin{aligned} (\dot{C}_3)^{\langle ab \rangle} = & -\Theta (C_3)^{ab} - \alpha D^{\langle a} (C_1)^{b \rangle} - 4(\alpha - \frac{1}{12}) \dot{n}^{\langle a} (C_1)^{b \rangle} \\ & - (\alpha + \frac{1}{3}) [\Theta D^{\langle a} \dot{n}^{b \rangle} + \frac{3}{2} \mu D^{\langle a} v^{b \rangle} + \frac{3}{2} v^{\langle a} D^{b \rangle} \mu] - 3(\alpha - \frac{1}{9}) \Theta \dot{n}^{\langle a} \dot{n}^{b \rangle} \\ & - 6(\alpha - \frac{1}{3}) \mu \dot{n}^{\langle a} v^{b \rangle} + 2\mu v_c v^{\langle a} [D^{\langle b \rangle} v^{c \rangle} + D^{[b \rangle} v^{c]}] \\ & + [(1 - \frac{1}{3} v^2) \Theta + \mu^{-1} (v^c D_c \mu) + \frac{5}{3} (D_c v^c)] \mu v^{\langle a} v^{b \rangle} \end{aligned} \quad (40)$$

$$(\dot{C}_4)^{\langle a \rangle} = -\frac{4}{3} \Theta (C_4)^a + \frac{1}{3} (1 + v^2) \mu (C_1)^a - \frac{1}{6} \mu v^a [v_b (C_1)^b] - E^a_b (C_1)^b + \frac{1}{3} \mu (C_6)^a \quad (41)$$

$$\begin{aligned} (\dot{C}_5)^{\langle a \rangle} = & (\alpha - 1) \Theta (C_5)^a + \alpha [D^a + 2 \dot{n}^a] [D_b (C_1)^b] + 4(\alpha - \frac{1}{12}) [D^a + 2 \dot{n}^a] [\dot{n}_b (C_1)^b] \\ & + 2 [(\alpha - \frac{1}{3}) D_b \dot{n}^b + 2(\alpha - \frac{1}{12}) (\dot{n}_b \dot{n}^b) - \alpha \frac{1}{6} \Theta^2 \\ & + \frac{1}{2} (\alpha + \frac{1}{2} (\alpha - 2) v^2 - \frac{1}{6} v^4) \mu] (C_1)^a \\ & + \frac{3}{2} (\alpha - \frac{5}{3} + \frac{1}{3} v^2) \mu v^a [v_b (C_1)^b] + \frac{3}{2} (\alpha + \frac{1}{3} + \frac{5}{9} v^2) \mu D^a (D_b v^b) \\ & + \frac{1}{2} (\alpha + \frac{1}{3} + (\alpha + \frac{4}{3}) v^2 - \frac{1}{3} v^4) \Theta D^a \mu - (\alpha + \frac{1}{3}) \Theta^2 \mu v^a \\ & + \frac{3}{2} (\alpha + \frac{1}{3} + (\alpha - \frac{16}{9}) v^2 + \frac{8}{9} v^4) \mu^2 v^a + \text{further non-linear terms} , \end{aligned} \quad (42)$$

where $(C_6)^a$ is defined by

$$\begin{aligned} (C_6)^a := & D^a (D_b v^b) - \frac{3}{2} D_b D^{\langle a} v^{b \rangle} \\ & - [\frac{1}{3} \Theta^2 - (1 - \frac{4}{3} v^2 + \frac{29}{12} v^4) \mu + \frac{10}{9} \Theta (D_b v^b) + 2 v^b D_b (D_c v^c) + \frac{3}{4} \mu^{-1} (D_b D^b \mu) \\ & + 3 (v_b D_c D^{[b} v^{c]}) - 3 (D_{[b} v_{c]} D^{[b} v^{c]}) + 3 (v^b \dot{n}^c D_{[b} v_{c]}) \\ & + 3 \mu^{-1} (v_b D^{[b} v^{c]} D_c \mu)] v^a \\ & + v_b [3 v^c D_c D^{[a} v^{b]} + \frac{3}{4} \mu^{-1} D^b D^a \mu + 5 (D_c v^c) D^{[a} v^{b]} + 3 \mu^{-1} (v^c D_c \mu) D^{[a} v^{b]} \\ & + 3 (v_c \dot{n}^c) D^{[a} v^{b]} + \frac{3}{2} D^{\langle a} \dot{n}^{b \rangle} + \frac{3}{2} \dot{n}^{\langle a} \dot{n}^{b \rangle}] \\ & + \frac{1}{2} (D_b v^b) \mu^{-1} D^a \mu - \frac{3}{4} D^{\langle a} v^{b \rangle} \mu^{-1} (D_b \mu) - \frac{1}{4} D^{[a} v^{b]} \mu^{-1} (D_b \mu) \\ & + 3 D^{\langle a} v^{b \rangle} D_{[b} v_{c]} v^c + 3 D^{[a} v^{b]} D_{\langle b} v_{c \rangle} v^c - 6 D^{[a} v^{b]} D_{[b} v_{c]} v^c . \end{aligned} \quad (43)$$

We remark that, besides $(C_1)^a$ to $(C_5)^a$, in principle also (C_G) and $(C_G)^{ab}$ given in Eqs. (30) and (31) are spatial constraints which need to be preserved along \mathbf{n} . However, the 1 + 3 covariant formulation does *not* provide natural evolution equations for the spatial curvature variables 3R and ${}^3S_{ab}$, or, rather, the underlying connection components relating to h_{ab} . To complete the constraint analysis, one instead needs to introduce either a 1 + 3 orthonormal frame (and the equations deriving from the Jacobi and Ricci identities listed in subsection A.4 of the appendix), or a set of local coordinates (see, e.g., Ref. [41]). Experience shows, nevertheless, that in general both (C_G) and $(C_G)^{ab}$ are preserved (cf., e.g., Refs. [24] and [41]).

Returning to the set (38) - (42), we now see that with the choice (36) both $(C_1)^a$ and $(C_2)^a$ will be preserved along \mathbf{n} , if each of $(C_1)^a$, $(C_2)^a$ and $(C_5)^a$ hold on an initial spacelike 3-surfaces $\mathcal{T}_0: \{t_0 = \text{const}\}$. However, propagation of $(C_4)^a$ generates the *secondary* constraint $(C_6)^a$.¹⁹ Furthermore, it is clear that neither $(C_3)^{ab}$ nor $(C_5)^a$ can be made to be preserved along \mathbf{n} for a *single* value of α , so they also generate secondary constraints. Hence, exact quasi-Newtonian cosmologies with dust matter source field

¹⁸Note the error in Eq. (B.13) of Ref. [21]: the numerical factor preceding the last term on the RHS should be ‘3’ rather than ‘2’.

¹⁹Note that the structure of $(C_6)^a$ is quite reminiscent of the general $D_b \sigma^{ab}$ -constraint (Ricci identity) in that it relates the spatial divergence of the shear $D_{\langle a} v_{b \rangle}$ of the peculiar velocity field v^a to the spatial gradient of its expansion $D_a v^a$.

and a lapse function evolution given by Eq. (36) are not consistent at the first derivative level as regards propagation of the constraints along \mathbf{n} — new conditions arise. The question is whether consistency can be attained at a higher derivative level, or with a prescription for \dot{N} alternative to Eq. (36); however, in view of the complexity which arises in the situation just discussed this seems doubtful except in very special cases. Indeed, initially our hope was that this project could lead to a substantial class of exact non-linear general relativistic generalisations of the Zel’dovich approximation method for gravitational instability scenarios; however, considering the complexity of the consistency equations, and noting the physical restrictions implied by excluding all gravitational radiation phenomena, we now believe there exist only very few consistent quasi-Newtonian cosmologies with dust matter source field.

We have been unable to resolve the generic consistency question by either 1 + 3 covariant or orthonormal frame methods. Thus, we end up with the open issue:

Problem 3: Determine (a) the set of cases where the constraint equations for a quasi-Newtonian cosmologies with dust matter source field are consistent with the time evolution equations, and (b) the related set of consistent quasi-Newtonian cosmologies.

In the following section we derive some of the consequences that arise from imposing particular dynamical restrictions on quasi-Newtonian cosmologies with dust matter source field. Thereafter, in section 6, we linearise the dynamical equations about a non-comoving FLRW background.

5 Constraint analysis in restricted cases

We have examined the reduced set of 1 + 3 covariant dynamical equations when particular geometrical restrictions are placed on the quasi-Newtonian cosmologies with dust matter source field, to see if we can solve the problem in those cases. In particular, we have examined quasi-Newtonian cosmologies with (a) zero spatial rotation of v^a , (b) zero spatial divergence of v^a , (c) LRS spacetime symmetry. In each case we find a more manageable reduced set of equations, but nevertheless have been unable to resolve the problem.

Where we have made more progress is in the cases of quasi-Newtonian cosmologies with (a) zero electric Weyl curvature (the conformally flat case), (b) the comoving case, and (c) zero relativistic acceleration. We now consider those cases in turn.

5.1 Zero E_{ab} — FLRW in a non-comoving frame

Specialising to zero electric Weyl curvature with respect to \mathbf{n} , it follows from Eq. (11) that

$$0 = E_{ab} = H_{ab} \quad \Leftrightarrow \quad C_{abcd} = 0 . \quad (44)$$

Now for $\mu \neq 0$ and $\Theta \neq 0$ it is well known that the *only* conformally flat non-static dust cosmologies belong to the FLRW family (see, e.g., Refs. [7, 12, 29, 35]). Thus, the expanding non-empty cosmological models of this kind are just the FLRW geometries as seen by a non-comoving Eulerian observer in a Newtonian-like rest frame (in this case these frames are not unique, because of the vanishing Weyl curvature, cf. the discussions in subsections 2.4 and 6.2.4).

From Eq. (22) we then find that $(C_4)^a$ becomes

$$(C_4)^a = -\frac{1}{3} (1 + \frac{1}{2} v^2) D^a \mu + \frac{1}{3} [\Theta + \frac{3}{2} \mu^{-1} (v^b D_b \mu) + \frac{5}{3} (D_b v^b)] \mu v^a + \frac{1}{6} \mu v_b [D^{(a} v^{b)} - 5 D^{[a} v^{b]}] , \quad (45)$$

(essentially: the equation $D_b E^{ab} = 0$), which can be treated either as an equation for $D_a \mu$ (determining the change of μ along the direction v^a in $\mathcal{T}: \{t = \text{const}\}$),²⁰ or for $D_a v_b$. Then, Eq. (18) provides the new constraint (essentially: the equation $\dot{E}^{(ab)} = 0$)

$$\begin{aligned} (C)^{ab} &:= \frac{2}{3} (1 - \frac{1}{4} v^2) \Theta \mu v^{(a} v^{b)} + \frac{1}{2} (v^c D_c \mu) v^{(a} v^{b)} - \frac{1}{2} v^{(a} D^{b)} \mu \\ &\quad + \frac{5}{6} (D_c v^c) \mu v^{(a} v^{b)} - \frac{1}{2} \mu D^{(a} v^{b)} + \mu v_c v^{(a} D^{(b)} v^{c)} + \mu v_c v^{(a} D^{(b)} v^{c]} , \end{aligned} \quad (46)$$

which is also an equation for $D_a \mu$ or $D_a v_b$. We will discuss the non-comoving FLRW case in linearised form in detail in section 6 below.

²⁰Even though this is a FLRW model, this will be non-zero if $v^a \neq 0$ because then we are in a tilted frame (cf. Ref. [28]).

5.2 Zero v^a — The comoving case

If we impose the comoving condition $v^a = 0$, it immediately follows from Eq. (17) that the relativistic acceleration vanishes:²¹

$$v^a = 0 \quad \Rightarrow \quad \dot{n}^a = 0 . \quad (47)$$

Then the equations simplify dramatically because by Eq. (21)

$$E_{ab} = 0 . \quad (48)$$

Thus, as condition (44) is satisfied, the only cosmologies satisfying Eq. (47) are the FLRW models in their standard (comoving) frame.

In detail: as $\dot{n}^a = 0$, we have by Eq. (4) that $N = N(t)$; the relativistic acceleration scalar potential is spatially constant,²² and we can choose $N = 1$ without loss of generality such that t in Eq. (5) then measures proper time of physical dimension [length] (see, e.g., Ref. [12]). From $(C_1)^a = 0$, $D_a \Theta = 0 \Rightarrow$ we can choose local coordinates so that $S = S(t)$ in Eq. (5). From $(C_2)^a = 0$, $D_a \mu = 0$, so Eq. (30) $\Rightarrow D_a {}^3R = 0$ and Eq. (31) $\Rightarrow {}^3S_{ab} = 0$. Hence, these cosmologies are spatially homogeneous (i.e., $\mu = \mu(t)$, $\Theta = \Theta(t)$, ${}^3R = {}^3R(t)$) with orthogonal spacelike 3-surfaces with 3-metric h_{ab} of constant curvature $K(t) := {}^3R/6$. Thus, the 3-metric $f_{\alpha\beta}(x^\gamma)$ in Eq. (5) can be chosen in one of the standard forms for a 3-space of constant dimensionless scalar curvature k where k can be normalised to one of $\{0, \pm 1\}$ by suitably renormalising $S(t)$ (see, e.g., Ref. [14]). The primary constraint equations are then all identically satisfied, and the remaining non-trivial propagation equations are Eqs. (15) and (16), with a first integral given by Eq. (30), provided $K(t) = k/S^2(t)$, which will indeed hold.

It is fundamentally important for what follows later, that by condition $(C_1)^a = 0$ the comoving case is precisely that case where the spatial gradient of the expansion rate is zero:

$$v^a = 0 \quad \Leftrightarrow \quad q^a = 0 \quad \Leftrightarrow \quad D^a \Theta = 0 . \quad (49)$$

Thus, we note that any of these conditions imply a comoving FLRW subcase of our quasi-Newtonian setting.

5.3 Zero \dot{n}^a — The geodesic case

When the relativistic acceleration of the shearfree normals \mathbf{n} vanishes, again

$$\dot{n}^a = 0 \quad \Leftrightarrow \quad N = N(t) ; \quad (50)$$

the preferred (Newtonian-like) Eulerian observers are in free fall, and so experience no effective gravitational field. The canonical choice for the lapse function is $N = 1$ (cf. subsection 5.2). Indeed, we have a zero-acceleration quasi-Newtonian cosmologies if and only if there are local coordinates in which the line element reduces to Eq. (5) with $N = 1$. Examples are Minkowski spacetime and the family of FLRW models. The question is whether there are other quasi-Newtonian cosmologies satisfying Eq. (50). For the conditions specified by the latter, Eq. (35), which does *not* assume a particular form for \dot{N} , only leads to an identity and so provides no new information. On the basis of the Ansatz (36), however, we find from Eq. (37) that for $\alpha \neq 0$, $\mu \neq 0$ and $(C_1)^a$ satisfied,

$$\dot{n}^a = 0 \quad \Rightarrow \quad v^a = 0 ; \quad (51)$$

this is just the comoving case just discussed in subsection 5.2.

6 Linearised perturbations of a FLRW dust cosmology

When discussing the evolution of *linear* spatially inhomogeneous deviations from a FLRW spacetime geometry, it is a common feature of many papers to introduce local coordinates and an explicit Ansatz for the infinitesimal line element (see, e.g., Refs. [3, 4, 36]). However, in doing so important physical

²¹We have imposed the condition that the matter is pressure-free (and so moves geodesically), and now that matter is moving along the integral curves of \mathbf{n} ; hence, these must be geodesic.

²²That is, it is constant in the preferred spacelike 3-surfaces \mathcal{T} : $\{t = \text{const}\}$ orthogonal to \mathbf{n} .

aspects of the dynamics of the setting often become rather obscured. We claim that it makes for clarity to work in $1 + 3$ covariant terms for as long as possible. In this section we discuss the important issue of a consistent Newtonian limit of FLRW-linearised cosmologies, once we have provided a $1 + 3$ covariant formulation of the setting at hand. For a comparison discussion, we first briefly review a recent treatment given by Bertschinger of the evolution of linear energy density inhomogeneities (i.e., so-called ‘scalar modes’) as induced by matter in *non*-relativistic motion [4]. The reason for referring to his work is that the timelike reference congruence he chooses is of the Newtonian-like kind, i.e., it is irrotational and shearfree. Other well-known discussions such as, e.g., that given by Peebles in Ref. [36] pick the timelike reference congruence to be irrotational and geodesic instead (in Bardeen’s terminology, the latter choice provides ‘synchronous’ gauge conditions [3]). We briefly refer to them at the end.

6.1 Bertschinger’s method

Bertschinger chooses what he calls ‘conformal Newtonian’ gauge conditions [4]. In Bardeen’s terminology these are ‘longitudinal’ gauge conditions, where the timelike normals \mathbf{n} to the spacelike 3-surfaces $\mathcal{T}: \{t = \text{const}\}$ are shearfree [3], and it is assumed that the matter perturbations do *not* produce any anisotropic pressures. The discussion in Ref. [4] then focuses on scalar linear spatially inhomogeneous deviations. Choosing a set of Cartesian spatial coordinates $\vec{x} := (x, y, z)$ comoving with \mathbf{n} on a spatially flat FLRW background as well as a conformally rescaled time variable η defined by²³

$$\frac{d}{dt} := \frac{1}{S} \frac{d}{d\eta} \quad \Rightarrow \quad \frac{d^2}{dt^2} = \frac{1}{S^2} \left[\frac{d^2}{d\eta^2} - \frac{1}{S} \frac{dS}{d\eta} \frac{d}{d\eta} \right], \quad (52)$$

it is stated that only one (scalar) variable, Φ , is needed to express the deviations in the background line element. Though no explicit explanation is given, this follows from the assumption of *zero* anisotropic pressure perturbations in combination with part of the linearised EFE [34]; we give the relevant details in subsection A.5 of the appendix. One so obtains the line element reduced to the simple form

$$ds^2 = S^2(\eta) [- (1 + 2\Phi) d\eta^2 + (1 - 2\Phi) (dx^2 + dy^2 + dz^2)]. \quad (53)$$

No further statement is made in Ref. [4] as to the η -dependence of Φ .

In the local coordinates employed, the proper peculiar velocity of a matter particle is given by $\vec{v} = d\vec{x}/d\eta$. It should here be stressed that in the Eulerian description adopted by Bertschinger the peculiar velocity “is defined as the average momentum per unit mass of the particles in the vicinity of a given Eulerian position” [4], i.e., in terms of the *momentum density* of the non-relativistically moving matter. Gravitational effects generated by the momentum density, however, are said to be neglected. Dynamical equations are now derived from the remaining linearised EFE on the basis of the assumptions that (1) the observable part of the Universe is nearly FLRW on scales of the present day Hubble radius, (2) $|\Phi| \ll 1$ and $v^2 \ll 1$, (3) only spatial inhomogeneities on scales much less than the present day Hubble radius are considered, and (4) the deviations in the energy density are dominated by non-relativistically moving matter only, so no anisotropic pressures need to be accounted for [4]. For dust, the set presented in Ref. [4] to evolve and constrain the deviations corresponds to

$$\delta\mu' = -3 \frac{S'}{S} \delta\mu - \mu_m \nabla \cdot \vec{v} - \nabla \cdot (\delta\mu \vec{v}) \quad (54)$$

$$\vec{v}' = -\frac{S'}{S} \vec{v} - (\vec{v} \cdot \nabla) \vec{v} - \nabla \Phi \quad (55)$$

$$0 = \nabla \cdot \nabla \Phi - \frac{1}{2} S^2 \delta\mu; \quad (56)$$

the prime denotes a partial derivative with respect to conformal time η , the ∇ -operator acts in terms of the comoving Cartesian coordinates \vec{x} , and the definition $\delta\mu := \mu - \mu_m$ was used (μ_m being the mean total energy density of the non-relativistic matter). Equation (54) is the continuity equation representing conservation of energy (density), Eq. (55) is the Euler equation representing conservation of momentum (density), while Eq. (56) is the PFE of Newtonian gravitation. In view of the Ansatz (53) for a spatially flat background, the PFE describes how matter inhomogeneities induce the irregularities in the almost-FLRW line element.

²³If, contrary to the time coordinate in Eq. (5), t here denotes dimensional proper time, then η is dimensionless.

Keeping only terms of first-order smallness, Bertschinger derives from the set (54) - (56) the standard second-order linear ODE [4, 36]

$$0 = \delta'' + \frac{S'}{S} \delta' - \frac{1}{2} S^2 \mu_m \delta \quad (57)$$

for the *dimensionless density contrast* $\delta := \delta\mu/\mu_m$. This is, however, a highly gauge and coordinate dependent procedure. By contrast, a corresponding equation can be derived in a more gauge-independent and covariant way by using geometrical variables and their dynamical equations. In the comoving case, this reproduces exactly the equation (57) (see Ref. [17]); however, in the non-comoving case corresponding to the situation discussed here, generically it gives a different result, as we now explain.

6.2 Covariant and gauge-invariant formulation

We contrast the description given by Bertschinger with a *non*-comoving 1+3 covariant and gauge-invariant formulation of linearised perturbations of FLRW models in the spirit of Ellis and Bruni [17], but in the quasi-Newtonian cosmology context. That is, (consistent with Eq. (53)) we assume the time slicing of $(\mathcal{M}, \mathbf{g}, \mathbf{n})$ provided by the spacelike 3-surfaces $\mathcal{T}: \{t = \text{const}\}$ with shearfree normals \mathbf{n} to have an approximately FLRW geometry, such that the quantities v^a and \dot{n}^a can be regarded as (gauge-invariant) deviations of *first*-order smallness. As we consider non-relativistically moving matter only, we *do* account for its momentum density q^a , which is *linear* in v^a , but *neglect* its isotropic and anisotropic pressures, p and π_{ab} , as they are of second-order smallness in v^a (cf. Eqs. (10)). The difference to the Ellis–Bruni treatment is that here we are *not* using a comoving choice of the timelike reference congruence, thus, the procedure is frame-independent iff the Newtonian-like timelike reference congruence \mathbf{n} is unique. We expect this to be the case in general, but, as discussed in subsection 2.4 above, there may be some freedom in its choice in specific restricted geometrical cases. In particular, we can view a strictly FLRW model from a tilted (non-comoving) reference frame. We will come back to the aspect of factoring out the pure non-comoving FLRW subcase, once we have investigated the general properties of the FLRW-linearised quasi-Newtonian dynamical equations.

6.2.1 The lapse function in almost-FLRW models

Given this context, we now return to the question as to how to evolve the lapse function N along the integral curves of our timelike reference congruence \mathbf{n} . The standard literature often deals with this aspect by merely stating (though generally not providing explicit mathematical relations; however, see, e.g., Refs. [10, 27, 38] for cases where the background is expanding, and Ref. [44] where it is not) that the non-relativistic peculiar motion of the matter induces only ‘slowly’ evolving variations in the metric structure of $(\mathcal{M}, \mathbf{g}, \mathbf{n})$. One way to interpret statements of this kind is to take $\dot{N} = 0$, but other options may also be taken into account. A number of possible ways of evolving the lapse function N along \mathbf{n} are contained in the Ansatz (36), which we continue to use in the subsequent analysis, leading to the evolution equation (37) for \dot{n}^a . Besides the ‘harmonic gauge’ given by $\alpha = 1$, this Ansatz also allows for the special subcases:

(i) $\alpha = -\frac{1}{3} \Rightarrow (SN)' = 0$; the dimensionless comoving (with the Eulerian observers) lapse function (SN) is covariantly constant along \mathbf{n} ; this choice is suggested from computing to first order the 1 + 3 ONF commutation functions associated with the line element (53): as can be derived from the relations given in subsection A.5 of the appendix one obtains $\dot{n}^\alpha = a^\alpha$, $0 = \sigma_{\alpha\beta} = \omega^\alpha = n_{\alpha\beta} = \Omega^\alpha$, and so the evolution equation for \dot{n}^α is given by Eq. (124) in subsection A.4.

(ii) $\alpha = 0 \Rightarrow \dot{N} = 0$; as mentioned above, here the lapse function N itself is covariantly constant along \mathbf{n} ;

(iii) $\alpha = \frac{1}{3}$; provided one makes the identification $N = S(1 + 2\Phi)^{1/2}$,²⁴ this value is implied by the analysis of *growing* mode ‘adiabatic’ perturbations in spatially flat FLRW backgrounds given by

²⁴At this point, one faces a problem with the physical dimension of the time coordinates involved. While our t in Eq. (5) is dimensionless, letting (as is standard in the ADM 3 + 1 formalism) N carry the dimension of the line element, in other works such as Refs. [34] and [27] t is dimensional proper time (and $N = 1$). This prevents an unambiguous identification of N with variables such as S and Φ , once a (dimensionless) conformal time coordinate η is introduced.

Mukhanov et al [34], p225, where (in their notation) $\Phi' = 0$.

6.2.2 Evolution of constraints

Besides v^a and \dot{n}^a , also the 1 + 3 covariant quantities $X_a := D_a \mu$ and $Z_a := D_a \Theta$ encode deviations of first-order smallness from an exact FLRW geometry of our time slicing; indeed we will regard X_a and Z_a as our prime perturbation variables. The evolution equations for these variables are

FLRW-linearised perturbation evolution equations:

$$\dot{X}^{(a)} = -\frac{4}{3}\Theta X^a - \mu Z^a - \mu D^a(D_b v^b) - \Theta \mu \dot{n}^a \quad (58)$$

$$\dot{Z}^{(a)} = -\Theta Z^a - \frac{1}{2}X^a + D^a(D_b \dot{n}^b) - \frac{1}{3}\Theta^2 \dot{n}^a - \frac{1}{2}\mu \dot{n}^a. \quad (59)$$

Note that these equations differ from Eqs. (49) and (50) for the comoving Lagrangean $\tilde{\mathbf{u}}$ -frame in Ellis and Bruni [17] by the presence of the second-order derivative terms as well as the terms algebraic in \dot{n}^a . Of importance in the following will also be the first-order evolution equations for the spatial divergences of both v^a and \dot{n}^a , given by

$$(D_a v^a)^\cdot = -\frac{2}{3}\Theta D_a v^a - D_a \dot{n}^a \quad (60)$$

$$(D_a \dot{n}^a)^\cdot = (\alpha - \frac{2}{3})\Theta D_a \dot{n}^a + \alpha \frac{3}{2}\mu D_a v^a + \alpha D_a (C_1)^a, \quad (61)$$

respectively.

In terms of X_a and Z_a the constraint equations (19) - (22), (29) and (43) now assume the first-order forms

FLRW-linearised constraint equations:

$$0 = (C_1)^a = Z^a - \frac{3}{2}\mu v^a \quad (62)$$

$$0 = (C_2)^a = \epsilon^{abc} D_b \dot{n}_c \quad (63)$$

$$0 = (C_3)^{ab} = E^{ab} - D^{(a} \dot{n}^{b)} \quad (64)$$

$$0 = (C_4)^a = D_b E^{ab} - \frac{1}{3}X^a + \frac{1}{3}\Theta \mu v^a \quad (65)$$

$$0 = (C_5)^a = D^a(D_b \dot{n}^b) - \frac{1}{2}X^a + \frac{1}{2}\Theta \mu v^a - \frac{1}{3}\Theta^2 \dot{n}^a + \mu \dot{n}^a \quad (66)$$

$$0 = (C_6)^a = D^a(D_b v^b) - \frac{3}{2}D_b D^{(a} v^{b)} - \frac{1}{3}\Theta^2 v^a + \mu v^a. \quad (67)$$

The momentum density of the moving matter $q^a = \mu v^a$ induces a spatial gradient Z^a in the expansion rate of the normals \mathbf{n} by $(C_1)^a$. Also note that in the given situation $E_{ab} = D_{(a} \dot{n}_{b)}$, provided $(C_3)^{ab}$ is satisfied and preserved along \mathbf{n} ; to first order the electric Weyl curvature of $(\mathcal{M}, \mathbf{g}, \mathbf{n})$ is completely determined by the distortion of \dot{n}^a . This relation reflects the connection between the tidal field and the gravitational scalar potential in the proper Newtonian theory [12]. We remark that in first-order approximation the 3-Cotton-York tensor of Eq. (32) is zero.

We now find that the set of constraints (62) - (67) evolves to first order according to

$$(\dot{C}_1)^{(a)} = -\Theta (C_1)^a + (C_5)^a \quad (68)$$

$$(\dot{C}_2)^{(a)} = (\alpha - \frac{2}{3})\Theta (C_2)^a \quad (69)$$

$$(\dot{C}_3)^{(ab)} = -\Theta (C_3)^{ab} - \alpha D^{(a} (C_1)^{b)} - (\alpha + \frac{1}{3})[\Theta D^{(a} \dot{n}^{b)} + \frac{3}{2}\mu D^{(a} v^{b)}] \quad (70)$$

$$(\dot{C}_4)^{(a)} = -\frac{4}{3}\Theta (C_4)^a + \frac{1}{3}\mu (C_1)^a + \frac{1}{3}\mu (C_6)^a \quad (71)$$

$$(\dot{C}_5)^{(a)} = (\alpha - 1)\Theta (C_5)^a + \alpha D^a (C_A) - \alpha \frac{1}{3}\Theta^2 (C_1)^a + (\alpha + \frac{1}{2})\mu (C_1)^a + \frac{3}{2}(\alpha + \frac{1}{3})\mu [D^a(D_b v^b) + \frac{1}{3}\Theta \mu^{-1} X^a - \frac{2}{3}\Theta^2 v^a + \mu v^a] \quad (72)$$

$$(\dot{C}_6)^{(a)} = -\Theta (C_6)^a - \frac{3}{2}D_b (C_3)^{ab} + \frac{3}{2}(C_4)^a - (C_5)^a \quad (73)$$

$$(\dot{C}_A) = -\frac{4}{3}\Theta (C_A) + D_a (C_5)^a, \quad (74)$$

where we have introduced the auxiliary constraint $(C_A) := D_a (C_1)^a$ in order to bring the evolution system (68) - (74) into a form suitable for application of the Cauchy-Kowalewskaya theorem on existence and

uniqueness of solutions for sets of *analytic* initial data (see, e.g., Ref. [44]). As can be immediately seen from Eqs. (70) and (72), we obtain the remarkable result that *the set of constraints (62) - (67) evolves consistently (without restrictions on the background model) for the special parameter value of $\alpha = -\frac{1}{3}$ in Eq. (36)*. As mentioned before, this is the situation when the dimensionless comoving lapse function (SN) is covariantly constant along \mathbf{n} .

If, however, $\alpha \neq -\frac{1}{3}$, further constraints arise, namely

$$0 = (C_7)^{ab} := \Theta D^{(a} \dot{n}^{b)} + \frac{3}{2} \mu D^{(a} v^{b)} \quad (75)$$

$$0 = (C_8)^{ab} := D^{(a} \dot{n}^{b)} + \frac{1}{2} \Theta D^{(a} v^{b)} \quad (76)$$

$$0 = (C_9)^{ab} := [(C_G) - {}^3R] D^{(a} v^{b)} , \quad (77)$$

the first arising from Eq. (70), and the latter two are each the condition resulting from consistency of the previous one. Indeed, these propagate along \mathbf{n} according to

$$(\dot{C}_7)^{(ab)} = (\alpha - 1) \Theta (C_7)^{ab} + \alpha \Theta D^{(a} (C_1)^{b)} - 2 \mu (C_8)^{ab} \quad (78)$$

$$(\dot{C}_8)^{(ab)} = (\alpha - \frac{7}{6}) \Theta (C_8)^{ab} + \alpha D^{(a} (C_1)^{b)} - \frac{3}{2} (\alpha - \frac{1}{6}) (C_9)^{ab} \quad (79)$$

$$(\dot{C}_9)^{(ab)} = -\frac{5}{6} \Theta (C_9)^{ab} - \frac{1}{2} [(C_G) - {}^3R] (C_8)^{ab} . \quad (80)$$

Finally, one can show that for $\alpha \neq -\frac{1}{3}$ the non-vanishing source term in the square bracket on the right-hand side of Eq. (72) can be expressed in terms of the other constraints, provided that ${}^3R = 0$:

$$(C_{10})^a := D_a (D_b v^b) + \frac{1}{3} \Theta \mu^{-1} D^a \mu - \frac{2}{3} \Theta^2 v^a + \mu v^a \quad (81)$$

$$= (C_6)^a + \Theta \mu^{-1} [D_b (C_3)^{ab} - (C_4)^a + D_b (C_8)^{ab}] - \frac{3}{4} [(C_G) - {}^3R] D_b D^{(a} v^{b)} . \quad (82)$$

Hence, for $\mu \neq 0$ and $\Theta \neq 0$ (and assuming that (C_G) is satisfied and preserved along \mathbf{n}), Eqs. (75) - (77) and (64) demand for $\alpha \neq -\frac{1}{3}$ that

$$0 = {}^3R D_{(a} v_{b)} \quad \Rightarrow \quad 0 = {}^3R E_{ab} ; \quad (83)$$

with this choice of evolution for N a complete and self-consistent set of variables and equations describing linearised scalar perturbations of FLRW dust cosmologies from a non-comoving point of view requires either

- a spatially flat background time slicing, ${}^3R = 0$, or
- a peculiar velocity field v^a with (to first order) vanishing shear, $D_{(a} v_{b)} = 0$. However, this is just the conformally flat case which we will further discuss below; the inhomogeneities are pure gauge in this case.

Thus, the conclusion is that *there are two consistent inhomogeneous FLRW-linearised cases: ${}^3R \neq 0$ and $\alpha = -\frac{1}{3}$, or ${}^3R = 0$ and no restriction on α . In each case the set of resulting equations is then consistent, but in the latter case there are the two extra constraints (75) and (76) that must be satisfied, as well as Eqs. (62) - (67), which must be satisfied in all cases.*

6.2.3 Evolution of spatially inhomogeneous deviations

It is now convenient to define scaled covariant and gauge-invariant perturbation variables by [17, 6]

$$\delta^a := \frac{SX^a}{\mu} , \quad \mathcal{Z}^a := SZ^a , \quad (84)$$

the first being the dimensionless comoving (with the Eulerian observers) fractional total energy density gradient, the second the comoving rate of expansion gradient. The purely *scalar* content of these variables is obtained by taking their magnitudes, or, essentially equivalently, by constructing their comoving spatial divergences [18, 6]:

$$\delta := SD_a \delta^a , \quad \mathcal{Z} := SD_a \mathcal{Z}^a . \quad (85)$$

In terms of the latter, Eqs. (58) and (59) convert into

$$\dot{\delta} = -\mathcal{Z} - SD_a[SD^a(D_b v^b)] - \Theta SD_a(S\dot{n}^a) \quad (86)$$

$$\dot{\mathcal{Z}} = -\frac{2}{3}\Theta\mathcal{Z} - \frac{1}{2}\mu\delta + SD_a[SD^a(D_b \dot{n}^b)] - \frac{1}{3}\Theta^2 SD_a(S\dot{n}^a) - \frac{1}{2}\mu SD_a(S\dot{n}^a), \quad (87)$$

which, finally, after some more work, can be combined into a single second-order linear ODE for δ given by

$$\ddot{\delta} + \frac{2}{3}\Theta\dot{\delta} - \frac{1}{2}\mu\delta = -\alpha\frac{3}{2}\Theta\mu SD_a(Sv^a) - \frac{1}{2}[2(\alpha - \frac{1}{3})\Theta^2 + (C_G) - {}^3R] SD_a(S\dot{n}^a) - \alpha\Theta SD_a[S(C_1)^a]. \quad (88)$$

In obtaining this relation, use was made of Eq. (30) and the fact that, with Eq. (37), we have to first order

$$[SD_a(S\dot{n}^a)]' = \alpha[\Theta SD_a(S\dot{n}^a) + \frac{3}{2}\mu SD_a(Sv^a) + SD_a[S(C_1)^a]]. \quad (89)$$

Equation (88) needs to be compared to results given in the standard literature (see, e.g., Refs. [3] and [36]), and the Bertschinger relation (57) given above. With $(C_1)^a$ and (C_G) satisfied initially and preserved along \mathbf{n} , *extra velocity-induced terms occur*. The most complex extra source term on the right-hand side will arise when $\alpha = -\frac{1}{3}$ which, as derived above, is the only case that allows for the possibility ${}^3R \neq 0$. However, even when ${}^3R = 0$, unless $\alpha = 0$ we have to account in particular for the first source term on the right-hand side of Eq. (88), which directly derives from the momentum density of the moving matter.

As Roy Maartens [31] kindly points out to us, in the special case when $\alpha = 0$ (corresponding to $\dot{N} = 0$) Eq. (88) can be brought into standard form (e.g., Eq. (11.1) in Ref. [36]) by using $0 = \mu - \frac{1}{3}\Theta^2$ and defining a new scalar perturbation variable $Y := \delta + 2SD_a(S\dot{n}^a)$; the second term then is covariantly constant along \mathbf{n} by Eq. (89). In this sense, computational simplicity singles out the value $\alpha = 0$ amongst all cases with a spatially flat background. The question is as to the physical content of this feature, since, unless ${}^3R \neq 0$, the value of α can be chosen completely arbitrarily. Solutions to the modified evolution equation (88) for δ can probably only be obtained by numerical means.

6.2.4 Tilted FLRW models

We now return to the special case considered above, where $D_{\langle a} v_{b \rangle} = 0$. This is just a FLRW model seen from a non-comoving frame. Indeed, from the non-comoving perspective, an Eulerian observer perceives spatial inhomogeneity even if the dust matter flow is *exactly* FLRW, i.e., if

$$0 = \dot{u}^a = \tilde{\sigma}_{ab} = \tilde{\omega}^a \quad \Rightarrow \quad 0 = \tilde{E}_{ab} = \tilde{H}_{ab},$$

because this is then a tilted spatially homogeneous cosmological model [28]. To first order in the deviations we find that if this tilted situation is given, then with $\mu \neq 0$ and $\Theta \neq 0$ it is characterised by:

(i) from Eq. (18)

$$0 = D_{\langle a} v_{b \rangle}, \quad (90)$$

(this is the condition that the spacetime is conformally flat to linear order); and

(ii) from Eq. (65), combined with Eqs. (62) and (30),

$$0 = (C_4)^a = -\frac{1}{6}D^a[{}^3R - (C_G)] - \frac{2}{9}\Theta(C_1)^a; \quad (91)$$

with $(C_1)^a$ and (C_G) satisfied, the intrinsic 3-Ricci curvature of the time slicing is found to be spatially homogeneous. From Eq. (67) we then have $0 = D^a(D_b v^b) + \frac{1}{2}{}^3R v^a$.

6.3 Derivation of the limiting Newtonian-like gravitational field equation

How does the difference between Eq. (88) and the Newtonian form (57) given by Bertschinger arise? The key role in standard treatments of FLRW-linearised matter and geometry perturbations is played by the PFE, the central equation in the Newtonian theory of gravitation, which, in our context, ties the spatial inhomogeneities in the total energy density distribution to the Laplacian of the relativistic acceleration scalar potential N . The problem seems to lie in the assumed exactly Newtonian form of this equation used by Bertschinger. We suggest that from the point of view of internal consistency this may not be the

appropriate Newtonian-like gravitational field equation to use in the cosmological context we consider here; that rather this equation should be the PFE modified by inclusion of a momentum density induced term.

Thus, the issue now is to discuss obtaining a consistent set of quasi-Newtonian dynamical equations in a cosmological context from the exact relativistic equations. Where can this come from?

6.3.1 The Raychaudhuri equation

The Raychaudhuri equation (15) becomes the relativistic generalisation of the PFE in the static case, when there is no expansion of the timelike reference congruence. Setting $\Theta = 0$ in Eq. (15) with $v^2 \ll 1$ gives the well-known gravi-static equation of gravitational attraction

$$D_a \dot{n}^a + (\dot{n}_a \dot{n}^a) = N^{-1} D_a D^a N = \frac{1}{2} \mu ,$$

— a direct generalisation of the Newtonian gravitational field equation.²⁵ The problem is that this does not work in the generic cosmological context: in a *non*-static cosmological spacetime this equation determines the rate of change of expansion of the model, with the spatial divergence of the relativistic acceleration as one of the source terms, rather than directly relating that divergence to the matter energy density.

The Raychaudhuri equation would become the relativistic generalisation of the PFE if we could suitably subtract out the rate of expansion, leaving a relation between energy density variation and the relativistic acceleration. The problem is that the background expansion is spatially homogeneous but the real expansion Θ is not, so subtracting off the background expansion terms is a gauge-dependent procedure. This will only give a gauge-invariant result if Θ is spatially homogeneous, i.e., if $Z^a = 0$. To make this work, we would have to subtract off the expansions terms not of a background model but of the real model; but they are spatially varying, and so would add extra terms to the PFE-like equation obtained. The resultant extra terms may be negligible in specific contexts (e.g. gravitational lensing models), but then that circumstance needs to be shown.

6.3.2 The Friedmann equation

The Friedmann equation for our quasi-Newtonian models takes the form

$$2\mu = {}^3R + \frac{2}{3}\Theta^2 .$$

If the background FLRW model is spatially flat, then with the line element in the form (53) this relation becomes

$$2\mu = \frac{4}{S^2} [\Phi_{|xx} + \Phi_{|yy} + \Phi_{|zz}] + \frac{2}{3}\Theta^2 .$$

Hence, again if we could subtract off the expansion term $\frac{2}{3}\Theta^2$ we would have an equation of the form we want. However, again that will not work in the generic physically relevant case when Θ is spatially inhomogeneous; we can subtract off the background expansion consistently in a position-independent way, but not the real expansion, for that is position-dependent. Again, this spatial dependence may be negligible in particular circumstances, but that needs to be shown (note that we are carrying out a consistent linearised study of the equations; it is not true that linearisation by itself implies we can neglect these terms).

What we really need to do is to take the spatial gradient of this equation to get a covariant and gauge-invariant equation of the form we want. However, we already have equations of that kind at hand: namely, the $D_b E^{ab}$ -constraint (Bianchi identity) and the compatibility condition $(C_5)^a$ binding the former to the shearfree slicing condition $(C_3)^{ab}$; we consider this next.

6.3.3 The $D_b E^{ab}$ -constraint (Bianchi identity)

In the case of a spatially flat background, the compatibility condition $(C_5)^a$ between the $D_b E^{ab}$ -constraint (Bianchi identity) $(C_4)^a$ and the zero-shear condition $(C_3)^{ab}$ assumes from Eq. (66) the form

$$D^a (D_b \dot{n}^b) = \frac{1}{2} X^a - \frac{1}{2} \Theta \mu v^a .$$

²⁵But generically with active gravitational mass $(\mu + 3p)$ (see, e.g., Whittaker [43] or Ehlers [7]).

In terms of the relativistic acceleration scalar potential N , this relation can be re-written as an effective Newtonian (PFE-like) gravitational field equation given by

$$D^a [D_b D^b N - \frac{1}{2} N \mu] = - \frac{1}{2} N \Theta \mu v^a ;$$

this is vectorial and third-order because it is covariant and gauge-invariant,²⁶ and it has an extra velocity part essentially resulting from the momentum constraint equation $(C_1)^a = 0$. We believe that any other approach that consistently derives a *scalar second-order PFE-like equation* in the current context will also lead to such an extra term.

6.3.4 Neglecting the momentum density

If (as, e.g., in Ref. [27]) one uses a version of any of these equations without the term proportional to $q^a = \mu v^a$, then one has neglected the first-order momentum density due to the non-relativistic peculiar motion of the matter. While this term is small, the implication, via $(C_1)^a = 0$, is that one has also neglected the spatial gradient Z^a of the expansion rate Θ of \mathbf{n} ; for example, the set (54) - (56) *neglects* this (0α) -EFE,²⁷ which links the matter momentum density — *linear* in \vec{v} — to mixed time-space derivatives of Φ (cf. Ref. [27] and subsection A.5 of the appendix).

To investigate this, we now inquire into the viability of imposing the *extra* assumption

$$q^a = \mu v^a = 0 \quad \Rightarrow \quad Z^a = 0 ;$$

thus, we set q^a to zero throughout, *implying*, as $\mu \neq 0$, that v^a vanishes. We here find ourselves in a slightly delicate situation, as, e.g, Bertschinger [4] neglects q^a but not v^a itself (part of his dynamical set is an evolution equation for \vec{v}). Note that, to first order, we have, e.g., $D_{\langle a} q_{b \rangle} = \mu D_{\langle a} v_{b \rangle}$ and $D_a q^a = \mu D_a v^a$.

From the gauge problem point of view, our extra assumption amounts to choosing a family of spacelike 3-surfaces \mathcal{T} : $\{t = \text{const}\}$ with timelike normals \mathbf{n} which, *simultaneously*, are shearfree and have spatially constant rate of expansion [3]. However, for $\mu \neq 0$ and $\Theta \neq 0$, it now follows that if $D_{\langle a} q_{b \rangle} = 0$, then, to first order,

$$D_{\langle a} v_{b \rangle} = 0 ,$$

which, for $v^a \neq 0$, we have recognised to characterise tilted FLRW models. In view of this result and the need to obtain *consistent* Newtonian limits from the exact equations of General Relativity in a cosmological context, enforcing the PFE by neglecting the first-order momentum density q^a of the matter in the (0α) -EFE seems questionable, even though this term is certainly very small.

It may be that other approaches, e.g., using ‘synchronous’ gauge conditions, get around this problem in a satisfactory way. However, for example Peebles’ approach to the CMBR anisotropies, that is largely based on this gauge, in fact mixes *different* gauges — setting up the CMBR anisotropy investigation in one gauge, and using another one to establish the PFE relating density inhomogeneities to the gravitational scalar potential (see Ref. [37], pp500-6). This seems a problematic procedure, and does not appear to solve the issue raised here.

7 Conclusion

Put in loose terms, in this article we have asked the question as to what extent we can squeeze a generic dust matter source field, which typically generates non-zero $\tilde{\Theta}$, $\tilde{\sigma}_{ab}$, $\tilde{\omega}^a$, \tilde{E}_{ab} and \tilde{H}_{ab} , into a Newtonian-like reference frame characterised by the conditions (1) - (4). The implied invariant physical requirement that a quasi-Newtonian cosmology $(\mathcal{M}, \mathbf{g}, \mathbf{n})$ excludes gravitational radiation provides a very strong restriction on the generality of the resulting cosmological model. It may turn out that this constraint proves to be sufficiently strong indeed that no exact spatially inhomogeneous solutions (in non-comoving description) exist at all. The complexity of the constraint analysis encountered in section 4 is certainly a reflection of the restrictiveness of the no-gravitational-radiation condition.²⁸ Indeed, it

²⁶Again, its purely scalar content is obtained from the spatial divergence of this relation.

²⁷The (0α) -EFE is equivalent to the $D_b \sigma^{ab}$ -constraint (Ricci identity) of the 1 + 3 covariant dynamical equations.

²⁸Note that this is different from other investigations of cosmological models without gravitational radiation, in particular those with $H_{ab} = 0$ [24], because in those cases the zero-magnetic-Weyl-curvature condition was imposed in the matter-comoving frame.

appears plausible that after eliminating the non-Newtonian spacetime curvature source terms (isotropic and anisotropic pressure) by FLRW-linearisation of the quasi-Newtonian equations, the gravitational radiation inducing quantities have been switched off, and the system of dynamical equations becomes tractable (and possibly solvable). If it could be shown that non-trivial solutions exist for the linear case but not the exact equations, linearisation instability of this dynamical setting would be shown.

Whether or not this is true, our analysis indicates problems arising in the Newtonian limit procedures in the literature based on quasi-Newtonian type conditions of the kind we discuss; use of mixed gauges as in Peebles' discussion [37] does not clarify these issues.

(1) The approximations made in the standard literature, ultimately leading to the PFE of the Newtonian theory of gravitation, cannot be regarded as providing an example of a *well*-defined (intrinsically consistent) Newtonian limit of the (cosmological) EFE, as demanded by Ehlers in, e.g., Refs. [8] and [9].

(2) If we use a covariant and gauge-invariant approach, this gives a setting where the ten general relativistic gravitational potentials reduce effectively to one, as in Newtonian theory (hence the common use of this approach in linearised studies). However, we still have unresolved the issue of what determines a suitable choice for time evolution of the $N(x^i)$ — we will get restricted results if we *impose* a choice for \tilde{N} ; but we have been unable to see how the integrability conditions generically *determine* the choice of $N(x^i)$ (which is what we expected). Thus, we do not have a solution in sight that satisfactorily resolves the problem. This corresponds to the unresolved question of how one obtains time-evolution for the Newtonian gravitational scalar potential with realistic cosmological boundary conditions.

(3) Factoring out the subset of (inhomogeneously perceived) spatially flat FLRW solutions in the linearised case characterised by the condition $D_{(a}v_{b)} = 0 \Rightarrow \tilde{\sigma}_{ab} = 0$,²⁹ we obtain extra velocity-dependent terms in the Newtonian-like gravitational field equations resulting from consistent linearisation of the relativistic equations, rather than just the PFE that many prefer to use.

(4) Underlying all this is the difficult issue of coarse graining: in essence we try to describe *two* distance scales (sub-Hubble-radius scales vs cosmological/spatial curvature relevant scales) by just *one* set of (mathematically consistent) dynamical equations. The relation between these different scales of analysis (and the corresponding effect on the EFE) needs to be clearly brought out and clarified.

Astrophysically oriented readers may immediately point out that our use of geometrised physical units disguises the true orders of magnitudes of the physical effects induced by various terms in our dynamical equations. For example, upon re-establishing the fundamental constants c and $8\pi G/c^2$ (in SI units we have $8\pi G/c^4 = 2.076 \times 10^{-43} \text{ s}^2/(kg m)$), the constraint $(C_1)^a$ of Eq. (62) reads

$$0 = D^a \Theta - \frac{12\pi G}{c^2} \rho v^a ,$$

where ρ denotes the mass density of the matter constituents. Similarly, non-standard source terms proportional to v^a in each of Eqs. (66) and (88) become quite small for non-relativistic magnitudes of v^a . Thus, in particular cases such as gravitational lensing settings, the extra terms may be negligible. Such arguments must, however, be pursued with care; as J L Synge has pointed out (see Ref. [39], pp176-8), if you too easily drop pressure gradient terms on the basis of similar arguments, you may deduce that an ocean liner will sink to the bottom of the sea. The argument needs to be carefully made in each context, for example, in Zel'dovich-like gravitational collapse scenarios. What we are pointing out is the necessity when doing so to look at the consistency of the whole set of equations, rather than looking at just one or two equations and ignoring the rest.

Acknowledgements

We are grateful to Bruce Bassett, Jürgen Ehlers and Roy Maartens for very helpful comments. This work has been supported by the South African Foundation for Research and Development (FRD) and the Deutsche Forschungsgemeinschaft (DFG). In parts the computer algebra package CLASSI was employed.

A Appendix

²⁹The spatial rotation $\epsilon^{abc} D_b v_c = 0 \Rightarrow \tilde{\omega}^a = 0$ is of *second*-order smallness in the first place.

A.1 Commutation relations

The following commutation relations for 1 + 3 covariantly defined scalars f and vectors V^a have been frequently used within our work:

$$h^b_a (D_b f)^\cdot = [D_a + \dot{n}_a] (\dot{f}) - \frac{1}{3} \Theta D_a f \quad (92)$$

$$D_{[a} D_{b]} V_c = E_{c[a} V_{b]} + \frac{1}{2} \mu v_c v_{[a} V_{b]} + h_{c[a} E_{b]d} V^d + \frac{1}{2} (v_d V^d) \mu h_{c[a} v_{b]} \quad (93)$$

$$+ \frac{1}{3} [(1 - v^2) \mu - \frac{1}{3} \Theta^2] h_{c[a} V_{b]} \quad (94)$$

$$D_{[a} D_{b]} V^b = -\frac{1}{2} E_{ab} V^b - \frac{1}{4} (v_b V^b) \mu v_a - \frac{1}{3} [(1 - \frac{1}{4} v^2) \mu - \frac{1}{3} \Theta^2] V_a . \quad (95)$$

Recall that in general the commutation relation for spatial derivative operators acting on f is given by $D_{[a} D_{b]} f = \epsilon_{abc} \omega^c f$ (cf. Ref. [18]). However, as for quasi-Newtonian cosmologies $\omega^a(\mathbf{n}) = 0$, within the $T: \{t = \text{const}\}$ we have

$$D_{[a} D_{b]} f = 0 . \quad (96)$$

A.2 Spatial derivatives of v^a

The irreducible decomposition of the purely spatial covariant derivative of the peculiar velocity v^a is given by

$$D_a v_b = D_{\langle a} v_{b\rangle} + \frac{1}{3} (D_c v^c) h_{ab} + \epsilon_{abc} [\frac{1}{2} \epsilon^{cde} D_d v_e] . \quad (97)$$

Now defining

$$p_{ab} := h_{ab} - v_a/v v_b/v , \quad q_{ab} := v_a/v v_b/v - \frac{1}{2} p_{ab} , \quad s_{ab} := \epsilon_{abc} v^c/v , \quad (98)$$

individual derivative terms can be expressed by

$$D_{\langle a} v_{b\rangle} = [2 v_{\langle a}/v p^{c}_{b\rangle} v^d/v + \frac{2}{3} q_{ab} q^{cd} + p^c_{\langle a} p^d_{b\rangle}] (D_c v_d) \quad (99)$$

$$D_a v^a = \frac{1}{2} v^a/v^2 D_a v^2 + p^a_b D_a v^b \quad (100)$$

$$\epsilon^{abc} D_b v_c = v^a/v (s^{bc} D_b v_c) + 2 s^{a[b} v^{c]} /v (D_b v_c) \quad (101)$$

$$D_a v^2 = v_a/v (v^b/v D_b v^2) + p^b_a D_b v^2 \quad (102)$$

$$v^b D_b v^a = \frac{1}{2} v^a/v (v^b/v D_b v^2) + p^a_b v^c D_c v^b . \quad (103)$$

Remember that the spatial rotation of v^a as given in Eq. (101) is algebraically constrained through Eq. (25).

A.3 Kinematical and Weyl curvature variables relating to the dust matter flow

The dust matter flow we discuss is moving with average 4-velocity $\tilde{\mathbf{u}}$; we have $\dot{\tilde{u}}^a = 0$. In deriving the following set of relations we made use of Eqs. (8), (17), (25) and (26):

$$\begin{aligned} \tilde{\sigma}_{ab}(v) = & \gamma [D_{\langle a} v_{b\rangle} + \frac{2}{3} \Theta n_{\langle a} v_{b\rangle} + \frac{1}{3} \Theta v^2 (n_a n_b + \frac{1}{3} h_{ab}) + n_{\langle a} v^c D_{|c|} v_{b\rangle}] \\ & + \gamma^3 [\frac{1}{2} (n_{\langle a} + v_{\langle a}) D_{b\rangle} v^2 + n_{\langle a} (n_b + v_b) (v^c v^d D_c v_d)] \\ & - \frac{1}{3} \gamma^3 [(1 - \frac{1}{3} v^2) \Theta + D_c v^c] [2 n_{\langle a} v_{b\rangle} + v^2 n_a n_b + v_a v_b] \end{aligned} \quad (104)$$

$$\tilde{\Theta}(v) = \gamma [(1 - \frac{1}{3} v^2) \Theta + D_a v^a] \quad (105)$$

$$\tilde{\omega}_{ab}(v) = \gamma [D_{[a} v_{b]} + n_{[a} v^c D_{|c|} v_{b]} - \frac{1}{2} \gamma^2 (n_{[a} + v_{[a} D_{b]} v^2 + \gamma^2 n_{[a} v_{b]} (v^c v^d D_c v_d)] . \quad (106)$$

As usual, the vorticity vector $\tilde{\omega}^a$ is given by

$$\tilde{\omega}^a = \frac{1}{2} \tilde{\epsilon}^{abc} \tilde{\omega}_{bc} . \quad (107)$$

In the non-comoving FLRW-linearised limit the kinematical variables of the dust matter flow are given by

$$\tilde{\sigma}_{ab} = D_{\langle a} v_{b \rangle} \quad (108)$$

$$\tilde{\Theta} = \Theta + D_a v^a \quad (109)$$

$$\tilde{\omega}_{ab} = 0; \quad (110)$$

recall that by Eq. (25) $D_{[a} v_{b]}$ is of second-order smallness.

The Weyl curvature variables with respect to the dust matter flow $\tilde{\mathbf{u}}$ can be obtained from

$$\tilde{E}_{ab}(\tilde{\mathbf{u}}) = -\dot{\tilde{\sigma}}_{\langle ab \rangle} - \frac{2}{3} \tilde{\Theta} \tilde{\sigma}_{ab} - \tilde{\sigma}^c_{\langle a} \tilde{\sigma}_{b \rangle c} - \tilde{\omega}_{\langle a} \tilde{\omega}_{b \rangle} \quad (111)$$

$$= \gamma^2 [(1 + v^2) E_{ab} + 2 (n_{\langle a} - v_{\langle a} \rangle E_{b \rangle c} v^c + (E_{cd} v^c v^d) (n_a n_b + v_a / v v_b / v + p_{ab})] \quad (112)$$

$$\tilde{H}_{ab}(\tilde{\mathbf{u}}) = -\tilde{D}_{\langle a} \tilde{\omega}_{b \rangle} + \tilde{\epsilon}_{cd \langle a} \tilde{D}^c \tilde{\sigma}^d_{b \rangle} \quad (113)$$

$$= 2 \gamma^2 \epsilon_{cd \langle a} [n_{b \rangle} (E^c_e v^e) + E^c_{b \rangle}] v^d. \quad (114)$$

A.4 1 + 3 orthonormal frame dynamical equations

For a full consistency analysis, we need the dynamical equations describing quasi-Newtonian cosmologies with dust matter source field expressed relative to a 1 + 3 orthonormal frame $\{\mathbf{e}_a\}$ (see, e.g., Refs. [11] and [23]). After aligning the timelike frame vector field \mathbf{e}_0 with our reference congruence \mathbf{n} , i.e., $\mathbf{e}_0 = \mathbf{n}$, one can choose a spatial frame $\{\mathbf{e}_\alpha\}$ according to convenience. Making use of this freedom, the following set of quasi-Newtonian relations, which we derive from the general equations given in Ref. [23] by simply deleting terms, can be further simplified.

Commutation relations:

$$[\mathbf{e}_0, \mathbf{e}_\alpha] = \dot{n}_\alpha \mathbf{e}_0 - \left[\frac{1}{3} \Theta \delta^\beta_\alpha - \epsilon^\beta_{\alpha\gamma} \Omega^\gamma \right] \mathbf{e}_\beta \quad (115)$$

$$[\mathbf{e}_\alpha, \mathbf{e}_\beta] = \left[2 a_{[\alpha} \delta^\gamma_{\beta]} + \epsilon_{\alpha\beta\delta} n^{\delta\gamma} \right] \mathbf{e}_\gamma. \quad (116)$$

Einstein field equations:

$$\mathbf{e}_0(\Theta) = -\frac{1}{3} \Theta^2 + (\mathbf{e}_\alpha + \dot{n}_\alpha - 2 a_\alpha) (\dot{n}^\alpha) - \frac{1}{2} (1 + v^2) \mu \quad (117)$$

$$0 = (\delta^{\gamma\langle\alpha} \mathbf{e}_\gamma + \dot{n}^{\langle\alpha} + a^{\langle\alpha} \rangle) (\dot{n}^{\beta\rangle}) + \mu v^{\langle\alpha} v^{\beta\rangle} - {}^*S^{\alpha\beta} - \epsilon^{\gamma\delta\langle\alpha} n^{\beta\rangle}_\gamma \dot{n}_\delta \quad (118)$$

$$0 = {}^*R + \frac{2}{3} \Theta - 2 \mu = (C_G) \quad (119)$$

$$0 = \delta^{\alpha\beta} \mathbf{e}_\beta(\Theta) - \frac{3}{2} \mu v^\alpha = (C_1)^\alpha, \quad (120)$$

where

$${}^*S_{\alpha\beta} := \mathbf{e}_{\langle\alpha} (a_{\beta\rangle}) + b_{\langle\alpha\beta\rangle} - \epsilon^{\gamma\delta}_{\langle\alpha} (\mathbf{e}_{|\gamma|} - 2 a_{|\gamma|}) (n_{\beta\rangle})_\delta \quad (121)$$

$${}^*R := 2 (2 \mathbf{e}_\alpha - 3 a_\alpha) (a^\alpha) - \frac{1}{2} b^\alpha_\alpha \quad (122)$$

$$b_{\alpha\beta} := 2 n_{\alpha\gamma} n^\gamma_\beta - n^\gamma_\gamma n_{\alpha\beta}. \quad (123)$$

Jacobi identities:

$$\mathbf{e}_0(a^\alpha) = -\frac{1}{3} (\delta^{\alpha\beta} \mathbf{e}_\beta + \dot{n}^\alpha + a^\alpha) (\Theta) + \frac{1}{2} \epsilon^{\alpha\beta\gamma} (\mathbf{e}_\beta + \dot{n}_\beta - 2 a_\beta) (\Omega_\gamma) \quad (124)$$

$$\mathbf{e}_0(n^{\alpha\beta}) = -\frac{1}{3} \Theta n^{\alpha\beta} + (\delta^{\gamma\langle\alpha} \mathbf{e}_\gamma + \dot{n}^{\langle\alpha} \rangle) (\Omega^{\beta\rangle}) - \delta^{\alpha\beta} (\mathbf{e}_\gamma + \dot{n}_\gamma) (\Omega^\gamma) - 2 \epsilon^{\gamma\delta\langle\alpha} n^{\beta\rangle}_\gamma \Omega_\delta \quad (125)$$

$$0 = \epsilon^{\alpha\beta\gamma} (\mathbf{e}_\beta - a_\beta) (\dot{n}_\gamma) - n^\alpha_\beta \dot{n}^\beta = (C_2)^\alpha \quad (126)$$

$$0 = (\mathbf{e}_\beta - 2 a_\beta) (n^{\alpha\beta}) + \epsilon^{\alpha\beta\gamma} \mathbf{e}_\beta(a_\gamma). \quad (127)$$

Electric Weyl curvature:

$$E_{\alpha\beta} = (\mathbf{e}_{\langle\alpha} + \dot{n}_{\langle\alpha} + a_{\langle\alpha} \rangle) (\dot{n}_{\beta\rangle}) + \frac{1}{2} \mu v_{\langle\alpha} v_{\beta\rangle} - \epsilon^{\gamma\delta}_{\langle\alpha} n_{\beta\rangle\gamma} \dot{n}_\delta + (C_3)_{\alpha\beta} \quad (128)$$

$$= {}^*S_{\alpha\beta} - \frac{1}{2} \mu v_{\langle\alpha} v_{\beta\rangle} - (C_G)_{\alpha\beta}. \quad (129)$$

Contracted Bianchi identities:

$$\mathbf{e}_0(\mu) = -(1 + \frac{1}{3}v^2) \Theta \mu - v^\alpha \mathbf{e}_\alpha(\mu) - \mu (\mathbf{e}_\alpha + 2\dot{n}_\alpha - 2a_\alpha)(v^\alpha) \quad (130)$$

$$\begin{aligned} \mathbf{e}_0(v^\alpha) = & -\frac{1}{3}(1 - v^2) \Theta v^\alpha - v^\beta (\mathbf{e}_\beta - \dot{n}_\beta - a_\beta)(v^\alpha) - v^2 a^\alpha - \dot{n}^\alpha \\ & + \epsilon^{\alpha\beta\gamma} [n_{\beta\delta} v^\delta v_\gamma + \Omega_\beta v_\gamma] . \end{aligned} \quad (131)$$

A.5 Line element and linearised Einstein field equations for scalar-perturbed spatially flat FLRW dust

Many perturbation calculations use the following quasi-Newtonian line element, where η denotes a dimensionless conformal time coordinate and x , y and z are Cartesian coordinates comoving with \mathbf{n} :

Infinitesimal line element:

$$ds^2 = S^2 [-(1 + 2\Phi) d\eta^2 + (1 - 2\Psi) (dx^2 + dy^2 + dz^2)] . \quad (132)$$

The associated commutation functions and components of the Einstein curvature tensor, as well as the first-order components of the energy-momentum-stress tensor are:

Commutation functions:

$$\dot{n}_\alpha = \frac{1}{S} \Phi_{|\alpha} \quad \Theta = \frac{3}{S} \left[(1 - \Phi) \frac{S_{|\eta}}{S} - \Psi_{|\eta} \right] \quad a_\alpha = \frac{1}{S} \Psi_{|\alpha} . \quad (133)$$

$$0 = \sigma_{\alpha\beta} = \omega^\alpha = \Omega^\alpha = n_{\alpha\beta} .$$

Einstein curvature tensor:

$$G_{00} = \frac{1}{S^2} \left[3(1 - 2\Phi) \left(\frac{S_{|\eta}}{S} \right)^2 - 6 \frac{S_\eta}{S} \Psi_{|\eta} + 2 \Psi^{| \alpha}_{|\alpha} \right] \quad (134)$$

$$G_{0\alpha} = \frac{2}{S^2} \left[\frac{S_{|\eta}}{S} \Phi_{|\alpha} + \Psi_{|\eta\alpha} \right] \quad (135)$$

$$\begin{aligned} G_{\alpha\alpha} = & -\frac{1}{S^2} \left[2(1 - 2\Phi) \frac{S_{|\eta\eta}}{S} - (1 - 2\Phi) \left(\frac{S_{|\eta}}{S} \right)^2 \right. \\ & \left. - 2 \frac{S_{|\eta}}{S} (\Phi + 2\Psi)_{|\eta} - 2 \Psi_{|\eta\eta} - (\Phi - \Psi)_{|\beta\beta} - (\Phi - \Psi)_{|\gamma\gamma} \right] \end{aligned} \quad \text{no summation, } \alpha \neq \beta \neq \gamma \quad (136)$$

$$G_{\alpha\beta} = -\frac{1}{S^2} (\Phi - \Psi)_{|\alpha\beta} \quad \alpha \neq \beta . \quad (137)$$

Energy-momentum-stress tensor:

$$T_{00} = \mu \quad T_{0\alpha} = \mu v_\alpha \quad T_{\alpha\beta} = 0 . \quad (138)$$

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